Test 2 for ISA5305, Fall 2019

Due in class of Thursday, December 26, 2019

Name : ________________ *ID* : ________________

(15%) 1(a) Let X be a continuous type of random variable that takes only nonnegative values. For any $\beta > 0$, prove that

$$P(X \geq \beta) \leq \frac{E(X)}{\beta}$$

(15%) 1(b) Let X be a continuous type of random variable with mean $E(X) = \mu$ and variance $Var(X) = \sigma^2$. For any $\tau > 0$, show that

$$P(|X - \mu| \geq \tau) \leq \frac{\sigma^2}{\tau^2}$$

(30%) 2. Fill the following blanks (no partial credits).

(a) Let $\{X_i,\ 1 \leq i \leq 9\}$ be a random sample of size 9 from $N(2,1)$. Define $\overline{X} = \frac{1}{9}\sum_{i=1}^{9} X_i$ and $Y = \sum_{i=1}^{9}(X_i - 2)^2$. Then

the moment-generating function $M_{\overline{X}}(t) =$ ______________________________

the moment-generating function $M_Y(t) =$ ______________________________

(b) Let $\{Z_i \sim N(0,1),\ 1 \leq i \leq 10\}$ be a random sample of size 10. Define $V = \sum_{i=1}^{10} Z_i$ and $W = \sum_{i=1}^{10} Z_i^2$. Then

the probability density function of V, $f_V(x) =$ ______________________________

the moment-generating function $M_W(t) =$ ______________ and the p.d.f. of W, $f_W(x) =$ ______________

(c) Let $\{X_1, X_2, \cdots, X_n\}$ be a random sample of size n from the *exponential distribution* with $E(X_n) = 3$. Define $Y = \sum_{i=1}^{n} X_i$. Then

the p.d.f. of Y, $f_Y(y) =$ ______________________________

the moment-generating function of Y, $\phi_Y(t) =$ ____________________

(d) Let $Z \sim N(0,1)$. Define $Y = 3Z + 2$, then

the p.d.f. of Y, $f_Y(y) =$ ______________________________

the moment-generating function of Y, $\phi_Y(t) =$ ____________________

(e) Let $\{X_i \sim \chi^2(1),\ 1 \leq i \leq n\}$ be a random sample and define $W_n = (\sum_{i=1}^{n} X_i - n)/\sqrt{2n}$. According to the Central Limit Theorem,

the limiting distribution function of W_n $limit_{n\to\infty} P(W_n \leq w) =$ ____________________

the limiting moment-generating function is $limit_{n\to\infty} M_{W_n}(t) =$ ____________________

(10%) 3. Let the r.v. X have the p.d.f. $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$, where $\alpha, \beta > 0$ are known positive integers.

Find $E[X]$ and $Var[X]$.

Hint: $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ and $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

(20%) 4. Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be the order statistics of a random sample $\{X_1, X_2, \cdots, X_n\}$ from the uniform distribution $U(0,1)$.

(a) Find the probability density function of $X_{(1)}$.

(b) Use the results of **(a)** to find $E[X_{(1)}]$.

(c) Find the probability density function of $X_{(n)}$.

(d) Use the result of **(c)** to find $E[X_{(n)}]$.

(10%) 5. Let X have the p.d.f. $f(x) = \beta x^{\beta-1}, \quad 0 < x < 1$ for a given $\beta > 0$. Define $Y = -3\beta ln(X)$.

(a) Compute the distribution function of Y, $P(Y \leq y), \ for \ 0 < y < \infty$.

(b) **Derive** the moment-generating function $M_Y(t)$.

(c) Compute $E(Y)$ and $Var(Y)$.