Homework 2: Computing Eigenvalues/Eigenvectors

1. This exercise asks you to write a program based on Jacobi transformations (Givens rotations) to compute eigenvalues and corresponding eigenvectors of real symmetric matrices.

*(a) Prove or realize that all eigenvalues of a real symmetric matrix are real (*skip this problem*).

(b) Write a C/C++, Python, Java, or Matlab program to compute eigenvalues/eigenvectors with the precision up to 10^{-4} and test the following Toeplitz matrix $T_5(\rho = 0.6)$ and tridiagonal matrix B_5 of order 5.

(c) Verify your results using Matlab.

$$T_5 = \begin{bmatrix} 1.0 & 0.6 & 0.36 & 0.216 & 0.1296 \\ 0.6 & 1.0 & 0.6 & 0.36 & 0.216 \\ 0.36 & 0.6 & 1.0 & 0.6 & 0.36 \\ 0.216 & 0.36 & 0.6 & 1.0 & 0.6 \\ 0.1296 & 0.216 & 0.36 & 0.6 & 1.0 \end{bmatrix}, \quad B_5 = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 4 \end{bmatrix}$$

2. Implement the following simple version of QR iteration with shift for computing the eigenvalues of a general real matrix $A = [a_{ij}]$.

Repeat

(a) $\sigma = a_{nn}$

(b) Compute QR factorization $A - \sigma I = QR$

(c) $A \leftarrow RQ + \sigma I$

Until Convergence

Q1. What convergence test should you use?

Q2. Test your program on the following matrices A and C.

Q3. Test your program on the matrices T_5 and B_5 defined in Problem **1** as well as the matrices of $T_{128}(\rho = 0.6)$ and B_{128}.

$$A = \begin{bmatrix} 11 & -12 & 8 & -4 \\ 25 & -25 & 16 & -8 \\ 7 & -6 & 2 & 0 \\ -9 & 9 & -8 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & -3 & 2 & -1 \\ 12 & -12 & 10 & -5 \\ 15 & -15 & 14 & -7 \\ 6 & -6 & 6 & -3 \end{bmatrix}.$$

Let $T_n(\rho) \in R^{n \times n}$ be a Toeplitz matrix defined as

$$T_n = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \cdots & \rho^{n-2} & \rho^{n-1} \\ \rho & 1 & \rho & \rho^2 & \cdots & \rho^{n-3} & \rho^{n-2} \\ \rho^2 & \rho & 1 & \rho & \rho^2 & \cdots & \rho^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho^{n-2} & \rho^{n-3} & \cdots & \cdots & \rho & 1 & \rho \\ \rho^{n-1} & \rho^{n-2} & \cdots & \cdots & \rho^2 & \rho & 1 \end{bmatrix}$$

A tridiagonal matrix $B_n \in R^{n \times n}$ can be given below.

$$B_n = \begin{bmatrix} 4 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ -1 & 4 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 4 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & -1 & 4 & -1 \\ 0 & 0 & \cdots & \cdots & 0 & -1 & 4 \end{bmatrix}$$