

Homework 1: Solving $A\mathbf{x}=\mathbf{b}$, Determinants, and Orthogonality

This exercise requires you to write an *efficient* and *effective* Matlab, C/C++, or Java program to solve a linear system of equations $A\mathbf{x} = \mathbf{b}$ by the strategy of LU decomposition with *partial pivoting*.

- (1) Apply the Gaussian Elimination with Partial Pivoting method to solve the following equations.

$$4.00001x + 1.00000y + 2.00000z = 4.00001$$

$$10.00000x - 0.10000y + 3.00000z = 12.80000$$

$$5.00000x + 3.00000y + 1.00000z = 12.00000$$

(a) Write this equation as $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

- (b) Find $PA = LU$, where L is unit lower- Δ and U is upper- Δ (using either Matlab or C/C++/Java programs).

- (c) Apply Gaussian elimination and back substitution to solve $PA\mathbf{x} = P\mathbf{b}$.

- (d) Can you get the same solution without using the partial pivoting strategy for this problem?

- (2) The $n \times n$ Hilbert matrix $H_n = [h_{ij}]$ is defined by

$$h_{ij} = \frac{1}{i+j-1}, \quad i, j = 1, 2, \dots, n$$

H_n can be generated using the MATLAB function **hilb**(n). It is well known that the Hilbert matrix is notoriously ill-conditioned.

- (a) Generate H_n , and $\mathbf{x}_n = \mathbf{1}_n = \mathbf{ones}(n, 1)$ for $n = 8, 12, 16, 20, 24$. In each case, construct $H_n\mathbf{x}_n = \mathbf{b}_n$ so that $\mathbf{1}_n$ is the exact solution. Then, by the given H_n and \mathbf{b}_n , applying the LU decomposition with partial pivoting strategy to solve \mathbf{x}_n .

- (b) Find the determinant of each H_n in (a).
- (c) Compute the condition number with p-norm, $p=1,2,\infty$ for each H_n in (a).
- (d) Discuss your solutions.

(3) Give the following Matlab code.

```
A=[1, 2, 3; 2, 6, 10; 3, 14, 28];
b=[1; 0; -8];
format short
X=A\b
```

- (a) Find the *LU-decomposition* for A.
 - (b) What does the above Matlab code do?
 - (c) What is the output of X?
- (4) Suppose that a tridiagonal matrix $T \in R^{n \times n}$ is also diagonally dominant.
- (a) Give an efficient algorithm to do $T = LU$.
 - (b) How many floating-point operations (flops) are needed for your algorithm?
- (5) To further understand the properties of Householder matrices such as symmetry, orthogonality, and the determinant, implement the following steps in Matlab environments.

```
for n=3:6
    v=rand(n,1);
    u=v/norm(v,2);
    H=eye(n)-2*u*u';
    [norm(H'-H,1), det(H), norm(H'*H-eye(n),1)]
    x=ones(n,1);
    e1=zeros(n,1); e1(1,1)=1;
    v=x-norm(x,2)*e1;
    u=v/norm(v,2);
    H=eye(n)-2*u*u';
    y=H*x
end
```

- (6) Let $A = [a_{ij}] \in R^{n \times n}$ be defined as $a_{ii} = 2$, $a_{ij} = -1$ if $|i - j| = 1$, and 0 otherwise.
- (a) Write down A_2 , A_3 in their matrix forms, what are $|A_2|$ and $|A_3|$?
 - (b) Show that $|A_{n+1}| = 2|A_n| - |A_{n-1}|$ and compute $|A_n|$ in terms of n .
 - (c) Write Matlab codes to generate a matrix A_n and its determinant for each given n .