H1 : Linear Systems of Equations

(1) Let $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$.

(a) Find AB and BA.

(b) Find $(BA)^t$ and A^tB^t.

(c) Find $(BA)^{-1}$. Can you find $(AB)^{-1}$?

(2) A linear system of equations is given below.

$$\begin{array}{rcrcrcr} 2x & - & 2y & + & 3z & = & 5 \\ -2x & + & 3y & - & 4z & = & -6 \\ 4x & - & 3y & + & 7z & = & 11 \end{array}$$

(a) Write this equation as $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(b) What is the augmented matrix for this system?

(c) Apply Gaussian elimination and back substitution to solve $A\mathbf{x} = \mathbf{b}$.

(d) Find $A = LU$, where L is unit lower$-\Delta$ and $U$is upper$-\Delta$.

(3) Let $\mathbf{e}_j \in R^n$ be a unit vector with j-th component 1 and 0 otherwise, e.g., $\mathbf{e}_2 = [0, 1, 0]^t$ for $n = 3$. Let the matrices P, Q, R be defined as

$$P = I + 2\mathbf{e}_2\mathbf{e}_1^t, \quad Q = I - 3\mathbf{e}_3\mathbf{e}_1^t, \quad R = I + 4\mathbf{e}_3\mathbf{e}_2^t.$$

(a) Compute P^{-1}, Q^{-1}, and R^{-1}.

(b) Compute $R^{-1}Q^{-1}P^{-1}$ and $(RQP)^{-1}$, respectively.

(4) Given

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 7 \\ 1 & 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 0 & 1 \\ 1 & 3 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & 7 \\ -5 & 3 & 5 \end{bmatrix}.$$

(a) Find an elementary matrix E such that $EA = B$.

(b) Find an elementary matrix F such that $AF = C$.

(5) A linear system of equations is given below.

$$\begin{aligned} x + y &= 1 \\ x + y + 2z &= 3 \\ 2x + y - 2z &= 0 \end{aligned}$$

(a) Write this equation as $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(b) What is the augmented matrix for this system?

(c) Apply Gaussian elimination with Partial Pivoting, followed by back substitution to solve $A\mathbf{x} = \mathbf{b}$.

(d) Find a permutation matrix P such that $PA = LU$, where L is unit lower$-\Delta$ and $U$is upper$-\Delta$.

(6) Give the following Matlab code.

```
A=[1, 2, 3; 2, 6, 10; 3, 14, 28];
b=[1; 0; -8];
format short
X=A\b
```

(a) Find the *LU-decomposition* for A.

(b) What does the above Matlab code do?

(c) What is the output of X?

(7) Suppose that a *tridiagonal matrix* $T \in R^{n\times n}$ is also diagonally dominant.

(a) Give an efficient algorithm to do $T = LU$.

(b) How many floating-point operations (flops) are needed for your algorithm?

(8) Let $C, D \in R^{n\times n}$ be invertible. prove the following equalities (Exercise 243 on P.87).

(a) $(C^{-1} + D^{-1})^{-1} = C(C + D)^{-1}D$

(b) $(I + CD)^{-1}C = C(I + DC)^{-1}$

(c) $(C + DD^t)^{-1}D = C^{-1}D(I + D^tC^{-1}D)^{-1}$

Solutions for Exercise 1

(1) $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$.

(a) $AB = \begin{bmatrix} 3 & 12 & 6 \\ 5 & -2 & 8 \\ 4 & 5 & 7 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 10 \\ 13 & 7 \end{bmatrix}$

(b) $(BA)^t = \begin{bmatrix} 1 & 13 \\ 10 & 7 \end{bmatrix}$, $A^tB^t = (BA)^t)$, (c) $(BA)^{-1} = \frac{1}{123}\begin{bmatrix} -7 & 10 \\ 13 & -1 \end{bmatrix}$, $(AB)^{-1}$ does not exist.

(2) A linear system of equations is given below.

$$\begin{array}{rcrcrcl} 2x & - & 2y & + & 3z & = & 1 \\ -2x & + & 3y & - & 4z & = & 0 \\ 4x & - & 3y & + & 7z & = & 5 \end{array}$$

(a) $\begin{bmatrix} 2 & -2 & 3 \\ -2 & 3 & -4 \\ 4 & -3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$, (b) $\left[\begin{array}{ccc|c} 2 & -2 & 3 & 1 \\ -2 & 3 & -4 & 0 \\ 4 & -3 & 7 & 5 \end{array}\right]$

(d) $LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow$ (c) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(3) Let $\mathbf{e}_j \in R^n$ be a unit vector with j-th component 1 and 0 otherwise, e.g., $\mathbf{e}_2 = [0,1,0]^t$ for $n = 3$. Let the matrices P, Q, R be defined as

$$P = I + 2\mathbf{e}_2\mathbf{e}_1^t, \quad Q = I - 3\mathbf{e}_3\mathbf{e}_1^t, \quad R = I + 4\mathbf{e}_3\mathbf{e}_2^t$$

$$(a)\ \ P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad R^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$

$$(b)\ \ R^{-1}Q^{-1}P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 11 & -4 & 1 \end{bmatrix}, \quad P^{-1}Q^{-1}R^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}$$

$$(c)\ \ P = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

(4)

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5 $\mathbf{x} = [1, 0, 1]^t$.

Matlab Solutions

(1)

```
      A=[ 3, 0; -1, 2; 1, 1];
      B=[ 1, 4, 2; 3, 1, 5];
  (a) AB=A*B,   BA=B*A
  (b) BAt=(B*A)'
  (c) BAinv=inv(BA)
```

(3)

```
      e1=[1, 0, 0]';
      e2=[0, 1, 0]';
      e3=[0, 0, 1]';
  (a) Pinv=inv(P), Qinv=inv(Q), Rinv=inv(R)
  (b) Rinv*Qinv*Pinv; Pinv*Qinv*Rinv
  (c) P=eye(3)+2*e2*e1'
      Q=eye(3)-3*e3*e1'
      R=eye(3)+4*e3*e2'
  (d) PQRinv=inv(P*Q*R)
```

(5)

```
      A=[1,1,0; 1,1,2; 2,1,-2];
      b=[1;3;0];
      x=A\b
      so x=[1, 0, 1]'
```