Test 1: Linear Algebra for ISA5305

Due by 11:00 am November 5, 2018

Name : ____________________ *ID* : ____________________ *Group no.* : ______

(20 pts)(1) Let $A, B \in R^{n \times n}$ be unit $lower - \Delta$ matrices, show that $C = AB$ is also unit $lower - \Delta$.

(20 pts)(2) Let $\mathbf{w} \in R^n$ be a unit vector, that is, $\|\mathbf{w}\|_2 = 1$, and denote $\mathbf{x} \in R^n$ as $\mathbf{x} = [x_1, x_2, \cdots, x_n]^t$ and $\sigma = \|\mathbf{x}\|_2$. Define a Householder matrix $G = I - 2\mathbf{w}\mathbf{w}^t$.

(a) Show that G is *symmetric*, *orthogonal*, and $G^{-1} = G$.

(b) Let $\mathbf{v} = \mathbf{x} + \sigma\mathbf{e}_1$ and $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|_2}$, define $H = I - 2\mathbf{u}\mathbf{u}^t$. Show that $H\mathbf{x} = -\sigma\mathbf{e}_1$.

(20 pts)(3) Let $A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$.

(a) Find the eigenvalues λ_1 and λ_2 of matrix A and their corresponding *unit* eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

(b) Find the trace of A, tr(A), and the determinant of A, det(A).

(c) Let $U = [\mathbf{u}_1, \mathbf{u}_2]$, find $U^t A U$.

(d) Find the eigenvalues μ_1 and μ_2 of matrix A^{-1}.

(e) Find the trace of A^{-1} and the determinant of A^{-1}.

(20 pts)(4) Let $A \in R^{n\times n}$ be a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and corresponding orthonormal eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$. For each nonzero vector $\mathbf{x} \in R^n$, the Rayleigh quotient $\rho(\mathbf{x})$ is defined by

$$\rho(\mathbf{x}) = \frac{\langle A\mathbf{x}, \mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle}$$

(a) For $\mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{u}_i$ with $\sum_{i=1}^{n} c_i^2 = 1$, prove that $\rho(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i c_i^2$

(b) Show that $\lambda_n \leq \rho(\mathbf{x}) \leq \lambda_1$

(c) Show that for $\mathbf{x} \neq \mathbf{0}$, $Min\{\rho(\mathbf{x})\} = \lambda_n$ and $Max\{\rho(\mathbf{x})\} = \lambda_1$

(20 pts)(5) Randomly generate a 4 by 4 matrix A with each element being an integer in [0,100], and a 4-dimensional integer column vector **b** by using matlab commands

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A=fix(50*random('uniform',0,1,4,4))
b=fix(100*random('uniform',0,1,4,1))
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Give **simple Matlab commands** *to solve each of the following questions* for $a \sim l$ and provide the solution.

(a) List the matrix A and the vector **b**.

(b) Solve **x** for A**x**=**b**.

(c) Find the determinant of A.

(d) Find the rank of A.

(e) Find $\|A\|_1, \quad \|A\|_2, \quad \|A\|_\infty$, respectively.

(f) Find the characteristic polynomial of A.

(g) Find the eigenvalues and corresponding eigenvectors of C.

(h) Find the singular values of matrix A.

(i) Compute the eigenvalues of $A^t A$.

(j) Compute the QR-factorization for A.

(k) Solve **y** for $R\mathbf{y} = Q^t\mathbf{b}$, with $Q, R, \mathbf{b}$ obtained above.

(l) Compute $\|\mathbf{x} - \mathbf{y}\|_2$.