Solutions for Test 1: Linear Algebra for ISA5305

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(20 pts)(1) Let $A, B \in R^{n\times n}$ be unit $lower-\Delta$ matrices, show that $C = AB$ is also unit $lower-\Delta$.

(Proof) $C = [c_{ij}] = AB$ with

$$\begin{aligned} c_{ij} &= \textstyle\sum_{k=1}^{n} a_{ik}b_{kj} \\ &= \textstyle\sum_{k=1}^{j-1} a_{ik}b_{kj} + a_{ij}b_{jj} + \sum_{k=j+1}^{n} a_{ik}b_{kj} \end{aligned}$$

Then

$$c_{ij} = 0 \;\; for\ 1 \le i < j \le n \;\; and \;\; c_{ii} = 1 \; for\ 1 \le i \le n$$

(20 pts)(2) Let $\mathbf{w} \in R^n$ be a unit vector, that is, $\|\mathbf{w}\|_2 = 1$, and denote $\mathbf{x} \in R^n$ as $\mathbf{x} = [x_1, x_2, \cdots, x_n]^t$ and $\sigma = \|\mathbf{x}\|_2$. Define a Householder matrix $G = I - 2\mathbf{w}\mathbf{w}^t$.

(a) Show that G is *symmetric*, *orthogonal*, and $G^{-1} = G$.

(b) Let $\mathbf{v} = \mathbf{x} + \sigma\mathbf{e}_1$ and $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|_2}$, define $H = I - 2\mathbf{u}\mathbf{u}^t$. Show that $H\mathbf{x} = -\sigma\mathbf{e}_1$.

(Proof of (b)) $\|\mathbf{v}\|_2^2 = 2\sigma^2 + 2\sigma x_1 = 2\sigma(\sigma + x_1)$ and $\mathbf{v}^t\mathbf{x} = \sigma^2 + \sigma x_1 = \sigma(\sigma + x_1)$, then $H\mathbf{x} = \mathbf{x} - (\mathbf{x} + \sigma\mathbf{e}_1) = -\sigma\mathbf{e}_1$.

(20 pts) 3. Let $A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$.

(a) Find the eigenvalues λ_1 and λ_2 of matrix A and their corresponding *unit* eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

(b) Find the trace of A and the determinant of A.

(c) Let $U = [\mathbf{u}_1, \mathbf{u}_2]$, compute $U^t AU$.

(d) Find the eigenvalues μ_1 and μ_2 of matrix A^{-1}.

(e) Find the trace of A^{-1} and the determinant of A^{-1}.

Ans: $p(A) = (\lambda + 2)(\lambda + 4) = 0$.

(a) $\lambda_1 = -2$, $\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\lambda_2 = -4$, $\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and

(b) $tr(A) = \lambda_1 + \lambda_2 = -6$ and $det(A) = \lambda_1 \times \lambda_2 = 8$.

(c) $U = [\mathbf{u}_1, \mathbf{u}_2]$, $U^{-1}AU = U^t AU = diag(-2, -4)$.

(d) $\mu_1 = -\frac{1}{2}$ and $\mu_2 = -\frac{1}{4}$ for A^{-1}.

(e) $tr(A^{-1}) = -\frac{3}{4}$ and $det(A^{-1}) = \frac{1}{8}$.

B-Ans: $p(B) = (\lambda - 1)^2 = 0$.

(Ba) $\lambda_1 = \lambda_2 = 1$, $\mathbf{u}_1 = \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(Bb) $tr(B) = \lambda_1 + \lambda_2 = 2$ and $det(B) = \lambda_1 \times \lambda_2 = 1$.

(Bc) $U = [\mathbf{u}_1, \mathbf{u}_2]$, $U^t BU = [1,\ 1;\ 1,\ 1]$ and

(Bd) $\mu_1 = 1$ and $\mu_2 = 1$ for B^{-1}.

(Be) $tr(B^{-1}) = 2$ and $det(B^{-1}) = 1$.

(20 pts) 4. Let A be a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and corresponding orthonormal eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$. For each nonzero vector $\mathbf{x} \in R^n$, the Rayleigh quotient $\rho(\mathbf{x})$ is defined by

$$\rho(\mathbf{x}) = \frac{\langle A\mathbf{x}, \mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle}$$

(a) For $\mathbf{x} = \sum_{i=1}^n c_i \mathbf{u}_i$ with $\sum_{i=1}^n c_i^2 = 1$, prove that $\rho(\mathbf{x}) = \sum_{i=1}^n \lambda_i c_i^2$

(b) Show that $\lambda_n \leq \rho(\mathbf{x}) \leq \lambda_1$

(c) Show that for $\mathbf{x} \neq \mathbf{0}$, $Min\{\rho(\mathbf{x})\} = \lambda_n$ and $Max\{\rho(\mathbf{x})\} = \lambda_1$

(a) Proof:

$$\langle \mathbf{x}, \mathbf{x}\rangle = \mathbf{x}^t\mathbf{x} = (\sum_{i=1}^n c_i\mathbf{u}_i)^t(\sum_{j=1}^n c_j\mathbf{u}_j) = \sum_{i=1}^n\sum_{j=1}^n c_ic_j\mathbf{u}_i^t\mathbf{u}_j = \sum_{i=1}^n c_i^2 = 1$$

$$\langle A\mathbf{x}, \mathbf{x}\rangle = \mathbf{x}^tA\mathbf{x} = (\sum_{i=1}^n c_i\mathbf{u}_i)^t(\sum_{j=1}^n c_jA\mathbf{u}_j) = \sum_{i=1}^n\sum_{j=1}^n \lambda_jc_ic_j\mathbf{u}_i^t\mathbf{u}_j = \sum_{i=1}^n \lambda_ic_i^2$$

Then

$$\rho(\mathbf{x}) = \frac{\langle A\mathbf{x}, \mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle} = \sum_{i=1}^n \lambda_ic_i^2$$

(b) Proof:

$$\lambda_n \leq \lambda_i \leq \lambda_1, \quad \forall\ 1 \leq i \leq n, \quad then$$

$$\lambda_n = \sum_{i=1}^n \lambda_nc_i^2 \leq \sum_{i=1}^n \lambda_ic_i^2 \leq \sum_{i=1}^n \lambda_1c_i^2 = \lambda_1$$

(c) Proof: In (**a**), let $c_1 = 1$ and $c_j = 0,\ 2 \leq j \leq n$, then $\rho(\mathbf{x}) = \lambda_1$. By (**b**), $Max\{\rho(\mathbf{x})\} = \lambda_1$. Similarly, $Min\{\rho(\mathbf{x})\} = \lambda_n$

(20 pts)(5) Randomly generate a 4 by 4 matrix A with each element being an integer in [0,100], and a 4-dimensional integer column vector **b** by using matlab commands

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A=fix(50*random('uniform',0,1,4,4))
b=fix(100*random('uniform',0,1,4,1))
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Give **simple Matlab commands** *to solve each of the following questions* for $a \sim l$ and provide the solution.

(a) List the matrix A and the vector **b**.

(b) Solve **x** for A**x**=**b**. *Ans:* $\mathbf{x} = A\backslash\mathbf{b}$

(c) Find the determinant of A. *Ans: det(A)*

(d) Find the rank of A. *Ans: rank(A)*

(e) Find $\|A\|_1$, $\|A\|_2$, $\|A\|_\infty$, respectively. *Ans:* $[norm(A,1), norm(A,2), norm(A,inf)]$

(f) Find the characteristic polynomial of A. *Ans: poly(A)*

(g) Find the eigenvalues and corresponding eigenvectors of C. *Ans:* [V,D]=eig(C)

(h) Find the singular values of matrix A. *Ans:* [U S V]=svd(A)

(i) Compute the eigenvalues of A^tA. *Ans:* [U,D]=eig(A'*A)

(j) Compute the QR-factorization for A. *Ans:* [Q R]=qr(A)

(k) Solve **y** for $R\mathbf{y} = Q^t\mathbf{b}$, with $Q, R, \mathbf{b}$ obtained above. *Ans:* $\mathbf{y} = R\backslash(Q' * \mathbf{b})$

(l) Compute $\|\mathbf{x}-\mathbf{y}\|_2$. *Ans:* $norm(\mathbf{x}-\mathbf{y}, 2)$