Random Variables of The Discrete Type

Definition: Given a random experiment with an outcome space S, a function X that assigns to each element s in S one and only one real number $x = X(s)$ is called a *random variable* (r.v.). The space of X is referred to as the set of real numbers $\Omega = \{x : \ X(s) = x, \ s \in S\}$.

Definition: The *probability mass function* (p.m.f) f of a discrete r.v. X is a function that satisfies the following properties:

(a) $f(x) > 0$, $x \in \Omega$;

(b) $\sum_{x\in\Omega} f(x) = 1$;

(c) $P(Y \subset \Omega) = \sum_{x\in Y} f(x)$;

□ Mathematical Expectation

Definition: If f is the p.m.f. of the r.v. X of the discrete type with space Ω and if the summation $\sum_{x\in\Omega} u(x)f(x)$ exists, then the sum is called the *mathematical expectation*, or the expected value of the function $u(X)$, which is denoted by $E[u(X)]$, that is,

$$E[u(X)] = \sum_{x\in\Omega} u(x)f(x).$$

Theorem: The mathematical expectation E satisfies

(a) If c is a constant, $E[c] = c$.

(b) If c is a constant, and u is a function, then $E[cu(X)] = cE[u(X)]$.

(c) If c_1, c_2 are constants, and u_1, u_2 are functions, then $E[c_1u_1(X) + c_2u_2(X)] = c_1E[u_1(X)] + c_2E[u_2(X)]$.

Definition: The kth moment $m_k,\ k = 1, 2, \cdots$ of a random variable X is defined by the equation

$$m_k = E(X^k), \quad where\ k = 1, 2, \cdots$$

Then $E(X) = m_1$, and $Var(X) = E[(X - E(X))^2] = E(X^2) - [E(X)]^2 = m_2 - (m_1)^2$.

Example 1. The number of defects on a printed board is a r.v. X with p.m.f. given by

$$P(X = i) = \frac{\gamma}{i+1}, \quad for\ i = 0, 1, 2, 3, 4$$

(a) Show that the constant $\gamma = \frac{60}{137}$.

(b) Show that $E(X) = \frac{163}{137}$ and $Var(X) = \frac{33300}{18769}$.

Example 2. The number of cells (out of 100) that exhibit *chromosome aberrations* is a random variable X with pmf given by

$$P(X = i) = \frac{\beta(i+1)^2}{2^{i+1}}, \quad for\ i = 0, 1, 2, 3, 4, 5$$

(a) Show that the constant $\beta = \frac{32}{159}$.

(b) Show that $E(X) = \frac{390}{159}$ and $Var(X) = \frac{57462}{25281}$.

Bernoulli, Geometric, Binomial, and Poisson Distributions

Bernoulli Trials A r.v. X assuming only two values 0 and 1 with the probability P(X=1)=p and P(X=0)=q=$1-p$ is called a Bernoulli r.v. Each action is called a Bernoulli trial.

Geometric Distribution Consider a sequence of independent Bernoulli trials. A r.v. X with the probability of the first success (X=1) at the $x-th$ trial equals

$$\mathrm{f}(x) = \mathrm{P}(\mathrm{X}{=}x) = q^{x-1}p,\ x = 1, 2, \ldots$$

Binomial Distribution Consider a sequence of n independent Bernoulli trials. A r.v. X with the probability of exactly x successes is

$$\mathrm{f}(x) = \mathrm{P}(\mathrm{X}{=}x) = \mathrm{C(n,x)}p^x q^{n-x},\ x = 0, 1, \ldots, n.$$

Poisson Distribution A r.v. X has a Poisson distribution with parameter $\lambda > 0$ if

$$\mathrm{f}(x) = \mathrm{P}(\mathrm{X}{=}x) = (e^{-\lambda}\lambda^x)/(x!),\ x = 0, 1, \ldots$$

Approximate Poisson Process

For the number of changes that occurs in a given continuous interval, we have an *approximate Poisson process* with parameter $\lambda > 0$ if

(1) The number of changes occurring in nonoverlapping intervals are independent.

(2) The probability of *exactly one change* in a sufficient short interval of length Δ is approximated by $\lambda\Delta$.

(3) The probability of *two* or *more* changes in a sufficient short interval is *essentially* zero.

Let λ be fixed, and $\Delta = \frac{1}{n}$ with a large n.

$$\begin{aligned}
P(X = x) &= \binom{n}{x} (\tfrac{\lambda}{n})^x (1 - \tfrac{\lambda}{n})^{n-x} \\
&= \tfrac{n!}{(n-x)!x!} \tfrac{\lambda^x}{n^x} (1 - \tfrac{\lambda}{n})^n (1 - \tfrac{\lambda}{n})^{-x} \\
&= \tfrac{n}{n} \cdot \tfrac{n-1}{n} \cdots \tfrac{n-x+1}{n} \tfrac{\lambda^x}{x!} (1 - \tfrac{\lambda}{n})^n (1 - \tfrac{\lambda}{n})^{-x} \\
&= \tfrac{\lambda^x e^{-\lambda}}{x!} \quad as\ n \ \to \ \infty
\end{aligned}$$

Moment-Generating Functions

Definition: Let X be a r.v. of the discrete type with p.m.f. f and the sample space S. if there is an $h > 0$ such that

$$M(t) \equiv E(e^{tX}) = \sum_{x \in S} e^{tx} f(x)$$

exists and is finite for $t \in (-h,\ h)$, then the function *M(t)* is called the *moment-generating function (m.g.f.)* of X.

Remark: If the *m.g.f.* exists, there is one and only one distribution of probability associated with that *m.g.f.*

Binomial Distribution: For $X \sim b(n, p),\ \textit{and}\ p + q = 1$,

$$\begin{aligned} M(t) &= E(e^{tX}) = \sum_{x=0}^{n} e^{tx} \begin{pmatrix} n \\ x \end{pmatrix} p^x q^{n-x} \\ &= \sum_{x=0}^{n} \begin{pmatrix} n \\ x \end{pmatrix} (pe^t)^x q^{n-x} = (q + pe^t)^n \end{aligned}$$

$E(X) = M'(0) = np$ and $Var(X) = M''(0) - [M'(0)]^2 = np(1 - p)$.

Poisson Distribution: Let X have a Poisson distribution with mean λ, then

$$\begin{aligned} M(t) &= E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} \\ &= \sum_{x=0}^{\infty} e^{-\lambda} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} \cdot e^{\lambda e^t} = e^{\lambda(e^t - 1)} \end{aligned}$$

$E(X) = M'(0) = \lambda$ and $Var(X) = M''(0) - [M'(0)]^2 = \lambda$.

Mean, Variance, and Moment Function of Discrete Distributions

Bernoulli $f(x) = p^x(1-p)^{1-x}, \quad x = 0, 1$

$$M(t) = 1 - p + pe^t; \qquad \mu = p, \;\; \sigma^2 = p(1-p)$$

Binomial $f(x) = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n$

$b(n,p) \qquad M(t) = (1 - p + pe^t)^n; \qquad \mu = np, \;\; \sigma^2 = np(1-p)$

Geometric $f(x) = (1-p)^{x-1}p, \quad x = 1, 2, \ldots$

$$M(t) = \frac{pe^t}{1-(1-p)e^t}, \qquad t < -ln(1-p)$$

$$\mu = \frac{1}{p}, \;\; \sigma^2 = \frac{1-p}{p^2}$$

Hypergeometric $f(x) = \frac{C(N_1,x)C(N_2,n-x)}{C(N,n)}, \quad x \le n, x \le N_1, \, n - x \le N_2$

$$M(t) = \times$$

$$\mu = n\left(\frac{N_1}{N}\right), \;\; \sigma^2 = n\left(\frac{N_1}{N}\right)\left(\frac{N_2}{N}\right)\left(\frac{N-n}{N-1}\right)$$

Negative Binomial $f(x) = C(x-1, r-1)p^r(1-p)^{x-r}, \quad x = r, r+1, r+2, \ldots$

$$M(t) = \frac{(pe^t)^r}{[1-(1-p)e^t]^r} \qquad t < -ln(1-p)$$

$$\mu = \frac{r}{p}, \;\; \sigma^2 = \frac{r(1-p)}{p^2}$$

Poisson $f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$

$$M(t) = e^{\lambda(e^t-1)}; \qquad \mu = \lambda, \;\; \sigma^2 = \lambda$$

Uniform $f(x) = \frac{1}{m}, \quad x = 1, 2, \ldots$

$$M(t) = \frac{1}{m} \cdot \frac{e^t(1-e^{mt})}{1-e^t}; \qquad \mu = \frac{m+1}{2}, \;\; \sigma^2 = \frac{m^2-1}{12}$$

Some Examples

1. Compute the probability function of the r.v. X that records the sum of the faces of two dice.

Solution: The sample space $\Omega = \{(i,j)|\ 1 \leq i,j \leq 6\}$. The random variable X is the function $X(i,j) = i+j$ which takes the range $R = \{2, 3, \cdots, 12\}$ with the probability function listed as

The probabilities of sum of two faces in casting two dice

$X(i,j)=s$	2	3	4	5	6	7
$P(X=s)$	1/36	2/36	3/36	4/36	5/36	6/36
$X(i,j)=s$	8	9	10	11	12	
$P(X=s)$	5/36	4/36	3/36	2/36	1/36	

2. It is claimed that 15% of the chickens in a particular region have patent H5N1 infection. Suppose seven chickens are selected at random. Let X equal the number of chickens that are infected.

(a) Assuming independence, how is X distributed? $[X \sim b(7, 0.15)]$.

(b) $P(X=1) = \begin{pmatrix} 7 \\ 1 \end{pmatrix} (0.15)^1(0.85)^6.$

(c) $P(X \geq 2) = 1 - P(0) - P(1) = 1 - (0.85)^7 - \begin{pmatrix} 7 \\ 1 \end{pmatrix} (0.15)^1(0.85)^6.$

3. Let a r.v. X have a binomial distribution with mean 6 and variance 3.6. Find $P(X=4)$.

Solution: Since $X \sim b(n,p)$ with $np = 6$ and $npq = 3.6$, then $q = 0.6$, $p = 0.4$, and $n = 15$. Thus, $P(X=4) = \begin{pmatrix} 15 \\ 4 \end{pmatrix} (0.4)^4(0.6)^{11} \approx 0.1992.$

4. Let a r.v. X have a geometric distribution. Show that

$$P(X > k + j | X > k) = P(X > j), \quad where \;\; k, j \geq 0$$

We sometimes say that in this situation there has been loss of momory.

Solution: Let p be *the rate of success* in a geometric distribution. Then $P(X > j) = \sum_{r=j+1}^{\infty}(1-p)^{r-1}p = (1-p)^j$, thus

$$P(X > k + j | X > k) = \frac{P(X > k + j)}{P(X > k)} = \frac{(1-p)^{k+j}}{(1-p)^k} = (1-p)^j = P(X > j).$$

5. Let X have a Poisson distribution with a variance of 3, then $P(X = 2) = \frac{e^{-3}3^2}{2!} \approx 0.224$ and $P(X = 3) = \frac{e^{-3}3^3}{3!} \approx 0.224$.

6. Flaws in a certain type of drapery material appear on the average of one in 150 square feet. If we assume the Poisson distribution, find the probability of at most one flaw in 225 square feet.

Solution: Since $\lambda = 225/150 = 1.5$, then $P(X \leq 1) = \frac{e^{-1.5}(1.5)^0}{0!} + \frac{e^{-1.5}(1.5)^1}{1!} \approx 0.5578$

Random Variables of Continuous Types

♣ Random variables whose space are intervals or a union of intervals are said to be of the *continuous types*. The *p.d.f.* of a r.v. X of continuous type is an integrable function $f(x)$ satisfying

(a) $f(x) > 0, \ x \in R$

(b) $\int_R f(x)dx = 1$

(c) The probability of the event $X \in A$ is $P(A) = \int_A f(x)dx$

□ The cumulative distribution function (cdf) is defined as $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$.

□ The expectation is defined as $\mu = E[X] = \int_{-\infty}^{\infty} xf(x)dx$

□ The variance is defined as $\sigma^2 = Var(X) = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$

□ The moment generating function is defined as $M(t) = \phi(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$, $-h < t < h$ for some $h > 0$.

□ The (100α)th percentile is x_α such that $F(x_\alpha) = \int_{-\infty}^{x_\alpha} f(x)dx = \alpha$.

♢ *Example:* Let X be the distance in feet between bad records on a used tape with the p.d.f.

$$f(x) = \tfrac{1}{40}e^{-x/40}, \quad 0 \leq x < \infty$$

Then the probability that no bad records appear within the first 40 feet is

$$P(X > 40) = \int_{40}^{\infty} f(x)dx = e^{-1} = 0.368$$

Exponential, Normal, χ^2, and Gamma Distributions

Uniform $U(a,b)$ $f(x) = \frac{1}{b-a}$, $a \le x \le b$

Exponential $f(x) = \frac{1}{\theta} e^{-x/\theta}$, $0 < x < \infty$

Gamma $f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}$, $0 < x < \infty$

$\chi^2(r)$ **Chi-Square** $f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{(r/2)-1} e^{-x/2}$, $0 < x < \infty$

Beta Distribution $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$

$N(\mu, \sigma^2)$ **Normal** $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/(2\sigma^2)}$, $-\infty < x < \infty$

Let $Z \sim N(0,1)$, $X \sim \chi^2(n)$, $Y \sim \chi^2(m)$, define $T = \frac{Z}{\sqrt{\chi^2(n)/n}}$ and $F = \frac{\chi^2(n)/n}{\chi^2(m)/m}$, then

Student-t Distribution $f_T(t) = \frac{\Gamma((n+1)/2)}{\sqrt{\pi n}\Gamma(n/2)[1+(t^2/n)]^{(n+1)/2}}$, $\infty < t < \infty$

F Distribution $f_F(w) = \frac{\Gamma((n+m)/2)(n/m)^{n/2} w^{(n/2)-1}}{\Gamma(n/2)\Gamma(m/2)[1+nw/m]^{(n+m)/2}}$, $0 < w < \infty$

Gamma, Exponential, χ^2 Distributions

Consider an (approximate) *Posisson distribution* with mean (arrival rate) λ, let the r.v. X be the waiting time until the αth arrival occurs. Then the cumulative distribution of X can be expressed as

$$\begin{aligned} F(x) &= P(X \le x) = 1 - P(X > x) \\ &= 1 - P(fewer\ than\ \alpha\ arrivals\ in\ (0, x]) \\ &= 1 - \textstyle\sum_{k=0}^{\alpha-1} [e^{-\lambda x} (\lambda x)^k / (k!)] \end{aligned} \tag{1}$$

$$f(x) = F'(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, 0 < x < \infty, \quad \alpha > 0 \tag{2}$$

Let $\theta = 1/\lambda$, we have the p.d.f. of Gamma Distribution

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, 0 < x < \infty \tag{3}$$

For Gamma distribution, if $\alpha = 1$, we have the p.d.f. of Exponential Distribution

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad 0 < x < \infty \tag{4}$$

For Gamma distribution, if $\theta = 2$ and $\alpha = r/2$, where r is a positive integer, then we have the p.d.f. of $\chi^2(r)$ distribution with r degrees of freedom.

$$f(x) = \frac{1}{\Gamma(r/2) 2^{r/2}} x^{(r/2)-1} e^{-x/2}, \quad 0 < x < \infty \tag{5}$$

For Gamma Distribution,

$M(t) = 1/(1 - \theta t)^\alpha$, $\mu = \alpha\theta$, $\sigma^2 = \alpha\theta^2$

For Exponential Distribution,

$M(t) = 1/(1 - \theta t)$, $\mu = \theta$, $\sigma^2 = \theta^2$

For $\chi^2(r)$ Distribution,

$M(t) = 1/(1 - 2t)^{r/2}$, $\mu = r$, $\sigma^2 = 2r$

Moment-Generating Function for Exponential Distribution

(1) Exponential distribution: $f(x) = \frac{1}{\theta}e^{-x/\theta}, \quad x > 0,$

$$\int_0^\infty f(x)dx = \int_0^\infty \frac{1}{\theta}e^{-x/\theta}dx = -e^{-x/\theta} \mid_{x=0}^{\infty} = 1 \tag{6}$$

$$\begin{aligned} \phi(t) &= E[e^{tX}] &= \textstyle\int_0^\infty e^{tx}f(x)dx \\ && = \textstyle\int_0^\infty [e^{tx}\frac{1}{\theta}e^{-x/\theta}]dx \\ && = \textstyle\frac{-1}{1-\theta t}e^{-[(1/\theta)-t]x}\Big|_{x=0}^{\infty} \\ && = \textstyle\frac{1}{1-\theta t} \quad for\ t < \frac{1}{\theta} \end{aligned} \tag{7}$$

$$E[X] = \phi'(0) = \frac{\theta}{(1-\theta t)^2} \mid_{t=0} = \theta \tag{8}$$

$$Var(X) = \phi''(0) - [\phi'(0)]^2 = \frac{2\theta^2}{(1-\theta t)^3} \mid_{t=0} - \theta^2 = \theta^2 \tag{9}$$

Moment-Generating Function for $N(\mu, \sigma^2)$ Distribution

(2) Normal distribution: $X \sim N(\mu,\ \sigma^2)$, $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty,$

(2.1) $\gamma = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

Proof:

$$\begin{aligned} \gamma^2 &= \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right) &&= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy \\ &= \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} J_{r,\theta}(x,y) drd\theta &&= \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} rdrd\theta \\ &= \frac{\pi}{4} \end{aligned} \tag{10}$$

(2.2) Given $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dx$ for $x > 0$, then $\Gamma(x+1) = x\Gamma(x)$.

(2.3) $\Gamma(n+1) = n!$ for $n \geq 0$, where $0! = 1$, and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(2.4) $X \sim N(\mu,\ \sigma^2)$, then $\phi(t) = E[e^{tX}] = e^{\mu t + \frac{\sigma^2 t^2}{2}}$.

Proof:

$$\begin{aligned} \phi(t) &= \int_{-\infty}^\infty [e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}] dx \\ &= \int_{-\infty}^\infty \left\{ e^{tx} \frac{1}{\sqrt{2\pi}\sigma} e^{[-(x-(\mu+\sigma^2 t))^2 + (2\mu t\sigma^2 + \sigma^4 t^2)]/2\sigma^2} \right\} dx \\ &= e^{\mu t + \frac{\sigma^2 t^2}{2}} \int_{-\infty}^\infty [\frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2}] dy \\ &= e^{\mu t + \frac{\sigma^2 t^2}{2}} \end{aligned} \tag{11}$$

Moment-Generating Function for $\chi^2(r)$ Distribution

(3) $Z \sim N(0,1) \ \Rightarrow \ Y = Z^2 \sim \chi^2(1)$.

Proof:

$$F(y) = P[Y < y] = P[-\sqrt{y} < Z < \sqrt{y}] = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \tag{12}$$

$$f(y) = F'(y) = \frac{1}{\Gamma(\frac{1}{2})2^{1/2}} y^{-1/2} e^{-y/2}, \quad 0 < y < \infty \tag{13}$$

(3.1) $Y \sim \chi^2(1)$, then $\phi(t) = \frac{1}{\sqrt{1-2t}}$.

Proof:

$$\begin{aligned} \phi(t) &= \textstyle\int_0^\infty [e^{tx} \frac{1}{\Gamma(\frac{1}{2})2^{1/2}} x^{-1/2} e^{-x/2}] dx \\ &= \textstyle\frac{1}{\Gamma(\frac{1}{2})2^{1/2}} \int_0^\infty [e^{-(\frac{1}{2}-t)x} x^{-1/2}] dx \\ &= \textstyle\frac{1}{\sqrt{1-2t}} \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty [e^{-y} y^{-1/2}] dy \\ &= \textstyle\frac{1}{\sqrt{1-2t}} \end{aligned} \tag{14}$$

(3.2) Let $Y_j \sim \chi^2(1),\ 1 \le j \le r$, be independent χ^2 distribution with 1 degree of freedom. Define $Y = \sum_{j=1}^r Y_j$, then $Y \sim chi^2(r)$ has χ^2 distribution with r degrees of freedom with the p.d.f. and the moment-generating function given below.

$$f(y) = \frac{1}{\Gamma(r/2)2^{r/2}} y^{(r/2)-1} e^{-y/2}, \quad 0 < y < \infty \tag{15}$$

$$\phi(t) = 1/(1-2t)^{r/2}, \quad E[Y] = \phi'(0) = r, \quad Var(Y) = \phi''(0) - [\phi'(0)]^2 = 2r \tag{16}$$

Proof:

$$\begin{aligned} \phi_Y(t) &= E[exp(tY)] &&= E[exp(\textstyle\sum_{j=1}^r tY_j)] \\ &= \Pi_{j=1}^r E[exp(tY_j)] &&= \Pi_{j=1}^r \phi_{Y_j}(t) \\ &= \Pi_{j=1}^r \textstyle\frac{1}{\sqrt{1-2t}} &&= \textstyle\frac{1}{(1-2t)^{r/2}} \end{aligned} \tag{17}$$

Moment-Generating Functions for Gamma Distributions

(4) *Gamma*(α, θ) distribution: $f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \quad 0 < x < \infty$

(4.1) For Gamma distribution, if $\alpha = 1$, we have the p.d.f. of Exponential Distribution

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \quad 0 < x < \infty \tag{18}$$

(4.2) For Gamma distribution, if $\theta = 2$ and $\alpha = r/2$, where r is a positive integer, then we have the p.d.f. of χ^2 distribution with r degrees of freedom, denoted as $X \sim \chi^2(r)$.

$$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{(r/2)-1} e^{-x/2}, \quad 0 < x < \infty \tag{19}$$

(4.3) $\phi(t) = 1/(1-\theta t)^\alpha$, $\mu = \alpha\theta$, $\sigma^2 = \alpha\theta^2$

(4.4) For Exponential distribution, $\phi(t) = 1/(1-\theta t)$, $\mu = \theta$, $\sigma^2 = \theta^2$

(4.5) For $\chi^2(r)$ distribution, $\phi(t) = 1/(1-2t)^{r/2}$, $\mu = r$, $\sigma^2 = 2r$

Proof:

$$\begin{aligned} \phi(t) &= \int_0^\infty \left[e^{tx} \frac{1}{\Gamma(\alpha)\alpha^\theta} x^{\alpha-1} e^{-x/\theta} \right] dx \\ &= \frac{1}{\Gamma(\alpha)\alpha^\theta} \int_0^\infty \left(e^{-(\frac{1}{\theta}-t)x} x^{\alpha-1} \right) dx \\ &= \frac{1}{(1-\theta t)^\alpha} \frac{1}{\Gamma(\alpha)} \int_0^\infty [e^{-y} y^{\alpha-1}] dy \\ &= \frac{1}{(1-\theta t)^\alpha} \end{aligned} \tag{20}$$

Normal (Gaussian) Distributions

A normal distribution of r.v. $X \sim N(\mu, \sigma^2)$ has the p.d.f.

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \tag{21}$$

□ $M(t) = \phi(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

□ $(X - \mu)/\sigma \sim N(0, 1)$

□ $Z \sim N(0, 1) \;\Rightarrow\; Y = Z^2 \sim \chi^2(1)$

When $\mu = 0$, $\sigma = 1$, $X \sim N(0, 1)$ is said to have the *standard normal distribution.* The cumulative distribution is denoted as

♢ $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx, \quad -\infty < z < \infty$

Define $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ for $x > 0$ and let $\gamma = \int_0^\infty e^{-x^2} dx$, show that

(a) $\gamma = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

(b) Show that $\Gamma(\frac{1}{2}) = 2\gamma = \sqrt{\pi}$

(c) $\Gamma(x + 1) = x\Gamma(x),\; for\ x > 0,\; \Gamma(n) = (n - 1)!$ if $n \in N$.

Exercises for $N(\mu, \sigma^2)$, Gamma(α, θ), χ^2 Distributions

1. Define $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ for $x > 0$ and let $\gamma = \int_0^\infty e^{-x^2} dx$, show that

(a) $\gamma = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

(b) Show that $\Gamma(\frac{1}{2}) = 2\gamma = \sqrt{\pi}$

(c) $\Gamma(x+1) = x\Gamma(x),\ \ for\ x > 0,\ \ \Gamma(n) = (n-1)!$ if $n \in N$.

2. Let $X \sim N(10, 36)$, write down the pdf for X and compute

(a) P$(X > 5)$.

(b) P$(4 < X < 16)$.

(c) P$(X < 8)$.

(d) P$(X < 20)$.

(e) P$(X > 16)$.

3. Let $Z \sim N(0,1)$ and $Y = Z^2$, then Y is said to have a χ^2 distribution of 1 degree of freedom, denoted as $\chi^2(1)$.

(a) Show that $f_Y(y) = \frac{1}{\Gamma(1/2)2^{1/2}} y^{-1/2} e^{-y/2},\ \ 0 < y < \infty$

(b) Show that $\phi(t) = \frac{1}{(1-2t)^{1/2}},\ \ t < \frac{1}{2}$

(c) If $Y_1, Y_2, \cdots, Y_k \sim \chi^2(1)$ and $Y_1, Y_2, \cdots, Y_k$ are independent, define $W = \sum_{j=1}^k Y_j$, then $W \sim \chi^2(k)$.

(d) Show that $\phi_W(t) = \frac{1}{(1-2t)^{k/2}}$ and $f_W(x) = \frac{1}{\Gamma(k/2)2^{k/2}} x^{(k/2)-1} e^{-x/2},\ \ 0 < x < \infty$

(e) $Y \sim \chi^2(6)$ is a random variable with 6 degrees of freedom, write down the pdf for Y and compute P$(Y \leq 6)$ and P$(3 \leq Y \leq 9)$.

4. If W is an exponential distribution with mean 6, write down the pdf for W and compute

(a) P$(W < 6)$.

(b) P$(W > 18 \mid W > 12)$.

Joint p.d.f. and Independent Random Variables

♣ Let X and Y be two discrete r.v.'s and let R be the corresponding space of X and Y. The joint p.d.f. of $X = x$ and $Y = y$, denoted by $f(x, y) = P(X = x, Y = y)$, has the following properties:

(a) $0 \leq f(x, y) \leq 1, \quad f(x, y) \geq 0$ for $-\infty < x, y < \infty$.

(b) $\sum_{(x,y)\in R} f(x, y) = 1, \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy = 1$.

(c) $P[(X, Y) \in A] = \sum_{(x,y)\in A} f(x, y) \ (\int \int_A f(x, y)), \quad A \subset R$.

♣ The marginal p.d.f. of X is defined as $f_X(x) = \sum_y f(x, y) \ \ (\int_{-\infty}^{\infty} f(x, y)dy), \quad x \in R_x$.

♣ The marginal p.d.f. of Y is defined as $f_Y(y) = \sum_x f(x, y) \ \ (\int_{-\infty}^{\infty} f(x, y)dx), \quad y \in R_y$.

♣ The random variables X and Y are independent iff $f(x, y) \equiv f_X(x)f_Y(y)$ for $x \in R_x$, $y \in R_y$.

Example 1. $f(x, y) = (x + y)/21, \quad x = 1, 2, 3; \quad y = 1, 2,$ then X and Y are not independent.

Example 2. $f(x, y) = (xy^2)/30, \quad x = 1, 2, 3; \quad y = 1, 2,$ then X and Y are independent.

♢ The collection of n independent and identically distributed random variables X_1, X_2, $\ldots$, X_n, is called *a random sample of size n from the common distribution*, say, $X_j \sim N(0, 1),\ 1 \leq j \leq n$.