Review of Calculus and Matrix Theory

(1) Evaluate $\int_0^\infty x^m \frac{1}{\theta} e^{-x/\theta} dx$ for $m = 0, 1, 2$, respectively.

(2) Evaluate $\int_0^\pi x^n \frac{1}{2} \sin(x) dx$ for $n = 0, 1, 2$, respectively.

(3) Show that $limit_{n\to\infty}(1 + \frac{h}{n})^n = e^h$.

(4) Evaluate the following integrals:

(a) $\int_0^{2\pi} sin(kx)sin(mx)dx$, where $k, m \geq 1,\ k \neq m$.

(b) $\int_0^{2\pi} cos(kx)cos(mx)dx$, where $k, m \geq 1,\ k \neq m$.

(c) $\int_0^{2\pi} sin(kx)cos(mx)dx$, where $k, m \geq 1$.

(d) $\int_0^{2\pi} sin(kx)sin(kx)dx$, where $k \geq 1$.

(e) $\int_0^{2\pi} cos(mx)cos(mx)dx$, where $m \geq 1$.

(5) Let $f(x) = e^{-x} - x$. Show that there exists an $x_0 \in [0, 1]$ such that $f(x_0) = 0$.

(6) Let $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.

(a) Find the characteristic polynomial of matrix A.

(b) Find the eigenvalues and corresponding eigenvectors of matrix A.

(c) What are the singular values of A?

□ *Some Solutions*

(4) 0 for $(a),\ (b),\ (c)$ and π for $(d),\ (e)$.

(5) $f(0) = e^{-0} - 0 = 1 > 0$ and $f(1) = e^{-1} - 1 < 0$.

(6) $P_A(x) = x^2 + 4x + 3$, $\lambda(A) = \{-1,\ -3\}$, $\sigma(A) = \{3,\ 1\}$,

Gamma Function and Its Properties

(7) Define $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ for $x > 0$ and let $\gamma = \int_0^\infty e^{-x^2}dx$. Then

- **(a)** $\Gamma(x+1) = x\Gamma(x),\ \ for\ x > 0,\ \ \Gamma(n) = (n-1)!$ if $n \in N$, where $0! \equiv 1$
- **(b)** $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$
- **(c)** $\Gamma(\frac{1}{2}) = 2\gamma = \sqrt{\pi}$
- **(d)** $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$

Proof:
(a)

$$\begin{aligned}\Gamma(x+1) = \int_0^\infty e^{-t}t^x dt &= \int_0^\infty t^x d(-e^{-t}) \\ &= -t^x e^{-t}\ |_0^\infty + \int_0^\infty x e^{-t}t^{x-1}dt \\ &= x\Gamma(x)\end{aligned}$$

Thus $\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \cdots = (n-1)!$

(b) $\gamma^2 = \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy.$

Let

$$\begin{matrix} x = r\cos\theta, & 0 \le \theta \le 2\pi \\ y = r\sin\theta, & 0 \le r < \infty \end{matrix}\ ,\quad J_{r,\theta} = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$$

Then

$$\begin{aligned}\gamma^2 &= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy \\ &= \int_0^\infty \int_0^{\pi/2} e^{-r^2} J_{r,\theta} dr d\theta \\ &= \int_0^\infty \int_0^{2\pi} r e^{-r^2} dr d\theta \\ &= \frac{\pi}{4}\end{aligned}$$

Thus $\gamma = \frac{\sqrt{\pi}}{2}$.

(c) $\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t}t^{\frac{1}{2}-1}dt = \int_0^\infty e^{-x^2}x^{-1}dx^2 = 2\int_0^\infty e^{-x^2}dx = 2\gamma = \sqrt{\pi}.$

Topics in Linear Algebra

- The central problems of Linear Algebra are to study the properties of matrices and to investigate the solutions of systems of linear equations
- Given A and b, find x for Ax=b; i.e., solve Ax=b
- Given A, find λ (lambda) such that Ax=λ x

Matrices and Linear Systems of Equations

- Matrix notations and operations
- Elementary row operations
- Row-echelon form
- Matrix inverse
- LU-Decomposition
- Some special matrices

Determinants

- Definition of Determinant det(A) or |A|
- Cofactor and minor at (i,j)-position of A
- Properties of determinants
- Examples
- Applications

Vector Space and Linear Transform

- Vector space, subspace, Examples
- Null space, column space, row space of a matrix
- Solutions of m equations in n unknowns
- Spanning sets, linear independence, rank, basis, dimension
- Vector norms and matrix norms
- Linear transform

  Kernel and Image

  Projection, rotation, scaling

  Gauss transform

  Householder transform (elementary reflector)

  Jacobi transform (Givens' rotation)

* Affine transform is in general not a linear transform

Orthogonality

- Motivation – More intuitive
- Inner product and projection
- Orthogonal vectors and linear independence
- Orthogonal complement
- Projection and least squares approximation
- Orthonormal bases and orthogonal matrices
- Gram-Schmidt orthogonalization process
- QR factorization

Eigenvalues and Eigenvectors

- Definitions and Examples
- Properties of eigenvalues and eigenvectors
- Diagonalization of matrices
- A Markov process
- Differential equations and exp(A)
- Similarity transformation and triangularization
  Positive definition matrices
  Spectrum decomposition
  Singular Value Decomposition (SVD)
- Algebraic multiplicity and geometric multiplicity
- Minimal polynomials and Jordan blocks

Topics in Probability Theory

1. Fundamentals of Probability
2. Discrete and Continuous Distributions
3. Multivariate Distributions
4. Statistics of Random Samples
5. Parameter Estimation

Some Applications

1. Principal Component Analysis (PCA)
2. Linear Discriminant Analysis (LDA)
3. Digital Image Processing
4. Cluster Analysis
5. Data Mining
6. Machine Learning
7. Pattern Recognition