

# Vector Space and Linear Transform

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## Definition of A Vector Space

**Definition:** A vector space  $V$  (over  $R$ ) is a set on which the operations of addition  $\oplus$  and scalar multiplication  $\odot$  are defined. The set  $V$  associated with the operations of addition and scalar multiplication is said to form a *vector space* if the following axioms are satisfied.

$$\text{(A1)} \quad \mathbf{x} \oplus \mathbf{y} = \mathbf{y} \oplus \mathbf{x} \quad \forall \mathbf{x}, \mathbf{y} \in V$$

$$\text{(A2)} \quad (\mathbf{x} \oplus \mathbf{y}) \oplus \mathbf{z} = \mathbf{x} \oplus (\mathbf{y} \oplus \mathbf{z}) \quad \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V$$

$$\text{(A3)} \quad \exists \mathbf{0} \in V \text{ such that } \mathbf{x} \oplus \mathbf{0} = \mathbf{x} \quad \forall \mathbf{x} \in V$$

$$\text{(A4)} \quad \forall \mathbf{x} \in V, \exists -\mathbf{x} \in V \text{ such that } \mathbf{x} \oplus (-\mathbf{x}) = \mathbf{0}$$

$$\text{(A5)} \quad \alpha \odot (\mathbf{x} \oplus \mathbf{y}) = (\alpha \odot \mathbf{x}) \oplus (\alpha \odot \mathbf{y}), \quad \forall \alpha \in R \text{ and } \mathbf{x}, \mathbf{y} \in V$$

$$\text{(A6)} \quad (\alpha + \beta) \odot \mathbf{x} = (\alpha \odot \mathbf{x}) \oplus (\beta \odot \mathbf{x}), \quad \forall \alpha, \beta \in R \text{ and } \mathbf{x} \in V$$

$$\text{(A7)} \quad (\alpha \cdot \beta) \odot \mathbf{x} = \alpha \odot (\beta \odot \mathbf{x}), \quad \forall \alpha, \beta \in R \text{ and } \mathbf{x} \in V$$

$$\text{(A8)} \quad 1 \odot \mathbf{x} = \mathbf{x} \text{ for a } 1 \in R \text{ and } \forall \mathbf{x} \in V$$

### *Examples*

$$\text{(1)} \quad R^n \text{ (over } R), \text{ in particular, } n = 2, 3$$

$$\text{(2)} \quad C[a, b], \text{ for example, } C[0, 1]$$

$$\text{(3)} \quad P_n = \{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \mid a_j \in R\}$$

$$\text{(4)} \quad R^{m \times n} = \text{the set of all } m \text{ by } n \text{ real matrices}$$

## Exercises for Vector Space

1. Mark  $\bigcirc$  if the vector addition and scalar multiplication forms a vector space, otherwise mark  $\times$ .

( $\times$ )    **(a)** For  $(R^3, \oplus, \odot)$ , the set of all triples of real numbers  $[x, y, z]$  with the operations

$$[u, v, w] \oplus [x, y, z] = [u + x, v + y, w + z] \quad \text{and} \quad \alpha \odot [x, y, z] = [\alpha x, y, z]$$

( $\times$ )    **(b)** For  $(R^3, \oplus, \odot)$ , the set of all triples of real numbers  $[x, y, z]$  with the operations

$$[u, v, w] \oplus [x, y, z] = [u + x, v + y, w + z] \quad \text{and} \quad \alpha \odot [x, y, z] = [0, 0, 0]$$

( $\times$ )    **(c)** For  $(R^2, \oplus, \odot)$ , the set of all paris of real numbers  $[x, y]$  with the operations

$$[u, v] \oplus [x, y] = [u + x, v + y] \quad \text{and} \quad \alpha \odot [x, y] = [2\alpha x, 2\alpha y]$$

( $\times$ )    **(d)** For  $(R^2, \oplus, \odot)$ , the set of all paris of real numbers  $[x, y]$  with the operations

$$[u, v] \oplus [x, y] = [u + x + 1, v + y + 1] \quad \text{and} \quad \alpha \odot [x, y] = [\alpha x, \alpha y]$$

( $\bigcirc$ )    **(e)** For  $(V, \oplus, \odot)$ , where  $V = \{[1, y] | y \in R\}$ , the set of all paris of real numbers  $[1, y]$  with the operations

$$[1, x] \oplus [1, y] = [1, x + y] \quad \text{and} \quad \alpha \odot [1, y] = [1, \alpha y]$$

( $\bigcirc$ )    **(f)** For  $(V, \oplus, \odot)$ , where  $V = \{x \in R | x > 0\}$ ,  $\alpha \in R$ ,

$$x \oplus y = xy \quad \text{and} \quad \alpha \odot x = x^\alpha$$

( $\bigcirc$ )    **(g)** For  $(V, \oplus, \odot)$ , where  $V = \{a + bx | a, b \in R\}$ ,

$$(a + bx) \oplus (c + dx) = (a + c) + (b + d)x \quad \text{and} \quad \alpha \odot (c + dx) = (\alpha c) + (\alpha d)x$$

## Subspaces of Vector Space

**Definition:** A subspace  $U$  of a vector space  $V$  is a *nonempty* subset satisfying

$$\mathbf{x} \oplus \mathbf{y} \in U \text{ and } \alpha \odot \mathbf{x} \in U \quad \forall \mathbf{x}, \mathbf{y} \in U; \alpha \in R$$

*Examples*

The set of lower- $\Delta$  (upper- $\Delta$ ) matrices

The set of tridiagonal (diagonal, Hessenberg) matrices

Let  $A \in R^{m \times n}$ ,  $A = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n]$ , and  $A^t = [\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m]$ , then

$$\text{Null}(A) = \{\mathbf{x} \in R^n \mid A\mathbf{x} = \mathbf{0}\} \subset R^n \text{ (Nullspace)}$$

$$R(A) = \{\sum_{j=1}^n \alpha_j \mathbf{a}_j \mid \alpha_j \in R\} \subset R^m \text{ (Column space)}$$

$$R(A^t) = \{\sum_{i=1}^m \beta_i \mathbf{b}_i \mid \beta_i \in R\} \subset R^n \text{ (Row space)}$$

**Theorem:** The system  $A\mathbf{x} = \mathbf{b}$  is solvable iff the vector  $\mathbf{b}$  can be expressed as a linear combination of the columns of  $A$

$$A = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n]$$

$$A\mathbf{x} = \mathbf{b} \text{ iff } \sum_{i=1}^n x_i \mathbf{a}_i = \mathbf{b}$$

# Overdetermined, Underdetermined, Homogeneous Systems

$$\begin{array}{cccccc}
 a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
 a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
 & & \cdot & & \cdot & & \cdot & & \cdot \\
 & & \cdot & & \cdot & & \cdot & & \cdot \\
 a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
 \end{array}$$

**Definition:** A linear system is said to be *overdetermined* if there are more equations than unknowns ( $m > n$ ), *underdetermined* if  $m < n$ , *homogeneous* if  $b_i = 0, \forall 1 \leq i \leq m$ .

$$\begin{array}{lll}
 x + y = 1 & x + y = 3 & x + y = 2 \\
 (A) \quad x - y = 3 & (B) \quad x - y = 1 & (C) \quad 2x + 2y = 4 \\
 -x + 2y = -2 & 2x + y = 5 & -x - y = -2
 \end{array}$$

(A) has no solution, (B) has unique solution, (C) has infinitely many solutions

$$\begin{array}{ll}
 (D) \quad x + 2y + z = -1 & x + 2y + z = 5 \\
 2x + 4y + 2z = 3 & (E) \quad 2x - y + z = 3
 \end{array}$$

(D) has no solution, (E) has infinitely many solutions

## Solutions of $m$ Equations in $n$ Unknowns

**Theorem:**  $\forall A \in R^{m \times n}$ , there corresponds a permutation matrix  $P$ , a unit lower- $\Delta$  matrix  $L$ , and an  $m \times n$  upper trapezoidal matrix  $U$  such that  $PA = LU$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 6 \\ 2 & 5 & 6 & 8 \\ 1 & 3 & 4 & 5 \end{bmatrix} \Rightarrow PA = P_{34}P_{23}A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 1 & 3 & 4 & 5 \\ 1 & 2 & 3 & 6 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 1 & 3 & 4 & 5 \\ 1 & 2 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = LU$$

## Linear Span

**Definition:** Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  be vectors in a vector space  $V$ . A sum of the form  $\sum_{i=1}^n c_i \mathbf{v}_i$ , where  $c_i$ 's are scalars, is called a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ . The *linear span* is the set of all linear combinations of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  and is denoted by

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$$

□ In  $R^3$ ,  $\text{span}(\mathbf{e}_1, \mathbf{e}_2) = \{[a, b, 0]^t \mid a, b \in R\}$

□ The nullspace could be  $\text{span}([1, -2, 1, 0]^t, [-1, 1, 0, 1]^t)$ , where

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

□  $\text{Null}(A) = \text{span}([1, -2, 1, 0]^t, [-1, 1, 0, 1]^t)$

□  $\text{Null}(A) = \text{span}([1, -2, 1, 0]^t, [0, -1, 1, 1]^t)$

**Theorem:** If  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are elements of a vector space  $V$ ,  $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$  is a subspace of  $V$ .

**Proof:** Show that  $a\mathbf{u} + b\mathbf{v} \in V, \forall a, b \in R; \mathbf{u}, \mathbf{v} \in V$

## Spanning Sets

**Definition:** The set of  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is a *spanning set* for  $V$  iff each  $\mathbf{v} \in V$  can be written as a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ .

- (1)  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, [1, 2, 3]^t\}$  is a spanning set of  $R^3$ .
- (2)  $\{[1, 1, 1]^t, [1, 1, 0]^t, [1, 0, 0]^t\}$  is a spanning set of  $R^3$ .
- (3)  $\{[1, 0, 1]^t, [0, 1, 0]^t\}$  is not a spanning set of  $R^3$ .
- (4)  $\{[1, 2, 4]^t, [2, 1, 3]^t, [4, -1, 1]^t\}$  is not a spanning set of  $R^3$ .
- (5)  $\text{span}(1, x, x^2) = \text{span}(1 - x^2, x + 2, x^2)$ , where  $P_2 = \{ax^2 + bx + c \mid a, b, c \in R\}$



## Linear Independence

**Definition:** The vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  are said to be *linearly independent* if  $\sum_{i=1}^n c_i \mathbf{v}_i = \mathbf{0}$  implies that  $c_i = 0$  for  $1 \leq i \leq n$ . Otherwise, they are said to be linearly dependent.

$$\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha = \beta = 0$$

Let

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{bmatrix}$$

- (1) The column vectors of  $A$  are *linearly independent*.
- (2) The column vectors of  $B$  are *linearly dependent*.
- (3) The column vectors of  $C$  are *linearly dependent*.

**Theorem:** A set of  $n$  vectors in  $R^m$  must be linearly dependent if  $n > m$

## Basis and Dimension

**Definition:** A *basis* for a vector space is a set of vectors satisfying two properties: (1) it is linearly independent, (2) it spans the vector space.

- $\{\mathbf{e}_1, \mathbf{e}_2\}$  is not a basis for  $R^3$  since  $\text{span}(\mathbf{e}_1, \mathbf{e}_2) \neq R^3$
- The vectors  $[1, 0]^t, [0, 1]^t, [2, 1]^t$  spans  $R^2$  but are not linearly independent so it is not a basis for  $R^2$

**Definition:** Any two bases for a vector space  $V$  contain the same number of vectors. This number, shared by all bases and expresses the number of freedom of the space, is called the *dimension* of  $V$ .

**Theorem:** Suppose that  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$  and  $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$  are both bases for the same vector space  $S$ , then  $m = n$ .

**Theorem:** Any linearly independent set in a vector space  $V$  can be extended to a basis by adding more vectors if necessary. Any spanning set in  $V$  can be reduced to a basis by discarding vectors if necessary.

**Example:** Let  $A \in R^{64 \times 17}$  be a matrix of rank 11.

(1)  $6 = (17 - 11)$  independent vectors  $\mathbf{x}$  satisfy  $A\mathbf{x} = \mathbf{0}$

(2)  $53 = (64 - 11)$  independent vectors  $\mathbf{y}$  satisfy  $A^t\mathbf{y} = \mathbf{0}$

## The Rank of A Matrix

□ The rank of a matrix  $A \in R^{m \times n}$  can be defined as the number of linear independent columns. In Matlab command:

`rank(A)`

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -1 & 1 & 1 \\ 1 & 1 & 2 & -1 \end{bmatrix}$$

Suppose the Gaussian elimination reduces  $A\mathbf{x}=\mathbf{b}$  to  $U\mathbf{x}=\mathbf{c}$  with  $r$  pivots, i.e., the last  $m - r$  rows are zero. Then, there is a solution only if the last  $m - r$  components of  $\mathbf{c}$  are also zero. If  $m = r$ , there is always a solution. The general solution is the sum of a particular solution (with all free variables zero) and a homogeneous solution (with  $n - r$  free variables as independent parameters). If  $r = n$ , there are no free variables and the nullspace contains only  $\mathbf{x}=\mathbf{0}$ . The number  $r$  is called the rank of matrix  $A$ .

Suppose  $\mathbf{x}_p$  satisfies  $A\mathbf{x}_p = \mathbf{b}$  and  $\mathbf{x}_h$  satisfies  $A\mathbf{x}_h = \mathbf{0}$

Then  $\mathbf{x}_g = \mathbf{x}_p + \mathbf{x}_h$  satisfies  $A\mathbf{x}_g = A\mathbf{x}_p + A\mathbf{x}_h = \mathbf{b} + \mathbf{0} = \mathbf{b}$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

## Four Fundamental Subspaces from a Matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow E = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

matrix  $\Rightarrow$  row echelon form  $\Rightarrow$  reduced row echelon form

♣ **Fundamental Theorem of Linear Algebra:** Let  $A \in R^{m \times n}$  have rank  $r$ ,

- (1)  $R(A)$ : the column space of  $A$ ,  $\dim(R(A)) = r$
- (2)  $N(A)$ : the nullspace of  $A$ ,  $\dim(N(A)) = n - r$
- (3)  $R(A^t)$ : the row space of  $A$  (the column space of  $A^t$ ),  $\dim(R(A^t)) = r$
- (4)  $N(A^t)$ : the left nullspace of  $A$  (the column space of  $A^t$ ),  $\dim(N(A^t)) = m - r$

- $N(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$
- $R(A) = \{\sum_{j=1}^n t_j \mathbf{a}_j \mid A = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n]\}$

The row space of  $A$  has the same dimension  $r$  as the row space of  $U$  because  $R(A^t) = R(U^t)$ . The nullspace  $N(A)$  has dimension  $n - r$ .

- (1)  $\dim(R(A)) + \dim(N(A)) = r + (n - r) = n$
- (2)  $\dim(R(A^t)) + \dim(N(A^t)) = r + (m - r) = m$

*Example:*  $A \in R^{3 \times 4}$

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$\begin{aligned} \dim(R(A)) &= 2 \\ \dim(N(A)) &= 4 - 2 \\ \dim(R(A^t)) &= 2 \\ \dim(N(A^t)) &= 3 - 2 \end{aligned}$$

# Vector Norms

**Definition:** A vector norm on  $R^n$  is a function

$$\tau : R^n \rightarrow R^+ = \{x \geq 0 \mid x \in R\}$$

that satisfies

**(1)**  $\tau(\mathbf{x}) > 0 \quad \forall \mathbf{x} \neq \mathbf{0}, \quad \tau(\mathbf{0}) = 0$

**(2)**  $\tau(c\mathbf{x}) = |c|\tau(\mathbf{x}) \quad \forall c \in R, \quad \mathbf{x} \in R^n$

**(3)**  $\tau(\mathbf{x} + \mathbf{y}) \leq \tau(\mathbf{x}) + \tau(\mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in R^n$

*Hölder norm (p-norm)*  $\|\mathbf{x}\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$  for  $p \geq 1$ .

**(p=1)**  $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$  (Mahattan or City-block distance)

**(p=2)**  $\|\mathbf{x}\|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2}$  (Euclidean distance)

**(p=∞)**  $\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} \{|x_i|\}$  (∞-norm)

## Matrix Norms

**Definition:** A matrix norm on  $R^{m \times n}$  is a function

$$\tau : R^{m \times n} \rightarrow R^+ = \{x \geq 0 \mid x \in R\}$$

that satisfies

- (1)  $\tau(A) > 0 \quad \forall A \neq O, \tau(O) = 0$
- (2)  $\tau(cA) = |c|\tau(A) \quad \forall c \in R, A \in R^{m \times n}$
- (3)  $\tau(A + B) \leq \tau(A) + \tau(B) \quad \forall A, B \in R^{m \times n}$

*Consistency Property:*  $\tau(AB) \leq \tau(A)\tau(B) \quad \forall A, B$

- (a)  $\tau(A) = \max\{|a_{ij}| \mid 1 \leq i \leq m, 1 \leq j \leq n\}$
- (b)  $\|A\|_F = \left[ \sum_{i=1}^m \sum_{j=1}^n a_{ij}^2 \right]^{1/2}$  (Fröbenius norm)

**Subordinate Matrix Norm:**  $\|A\| = \max_{\|\mathbf{x}\| \neq 0} \{\|A\mathbf{x}\| / \|\mathbf{x}\|\}$

- (1) If  $A \in R^{m \times n}$ , then  $\|A\|_1 = \max_{1 \leq j \leq n} (\sum_{i=1}^m |a_{ij}|)$
- (2) If  $A \in R^{m \times n}$ , then  $\|A\|_\infty = \max_{1 \leq i \leq m} (\sum_{j=1}^n |a_{ij}|)$
- (3) Let  $A \in R^{n \times n}$  be real symmetric, then  $\|A\|_2 = \max_{1 \leq i \leq n} |\lambda_i|$ , where  $\lambda_i \in \lambda(A)$

# Linear Transformation

**Definition:** A mapping  $L$  from a vector space  $V$  to a vector space  $W$  is said to be a linear transform (transformation) or a linear operator if

$$L(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) = (\alpha \odot L(\mathbf{v}_1)) \oplus (\beta \odot L(\mathbf{v}_2)), \quad \forall \alpha, \beta \in R, \mathbf{v}_1, \mathbf{v}_2 \in V$$

*Examples: Projection, Scaling, Rotation, Reflection on  $V = R^2$*

(a)  $L(\mathbf{x}) = \mathbf{u}^t \mathbf{x}$ , for  $\mathbf{u} \in V$

(b)  $L(\mathbf{x}) = s\mathbf{x}$ , for  $s \in R$

(c)  $L(\mathbf{x}) = R_\theta \mathbf{x}$ , where  $R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

(d)  $L(\mathbf{x}) = \mathbf{y}$ , where  $y_1 = -x_1$  and  $y_2 = x_2$

(e)  $L(f) = F$ , where  $f \in C[a, b]$  and  $F(x) = \int_a^x f(t) dt$

(f)  $L(f) = f'$ , where  $f \in C^1[a, b]$  and  $f'(x) = \frac{d}{dx} f(x)$

(g)  $L(\mathbf{x}) = A\mathbf{x}$ , where  $\mathbf{x} \in R^n$ ,  $A \in R^{m \times n}$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

## Image and Kernel

Let  $L : V \rightarrow W$  be a linear transform, and let  $S \subset V$  be a *subspace* of  $V$ . The *kernel* of  $L$ , denoted by  $\text{Ker}(L)$ , is defined by

$$\text{Ker}(L) = \{\mathbf{v} \in V \mid L(\mathbf{v}) = \mathbf{0}\}$$

The *image* of  $S$  under  $L$ , denoted by  $L(S)$ , is defined by

$$L(S) = \{\mathbf{w} \in W \mid \mathbf{w} = L(\mathbf{v}) \text{ for some } \mathbf{v} \in S\}$$

**Theorem:** Let  $L : V \rightarrow W$  be a linear transform, and let  $S \subset V$  be a *subspace* of  $V$ , then

- (a)  $\text{Ker}(L)$  is a subspace of  $V$
- (b)  $L(S)$  is a subspace of  $W$



## Changing Coordinates in $R^2$

$$\{\mathbf{e}_1, \mathbf{e}_2\} \Rightarrow \{\mathbf{v}_1, \mathbf{v}_2\}$$

Any vector in  $\mathbf{w} \in R^2$  can be expressed as  $\mathbf{w} = x\mathbf{e}_1 + y\mathbf{e}_2 = [x, y]^t$ , suppose that we want to express  $\mathbf{w}$  as a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$  as  $\mathbf{w} = x'\mathbf{v}_1 + y'\mathbf{v}_2 = [x', y']^t$ . What are  $\{x, y\}$  and  $\{x', y'\}$  related?

$$\begin{bmatrix} x \\ y \end{bmatrix} = x\mathbf{e}_1 + y\mathbf{e}_2 = x'\mathbf{v}_1 + y'\mathbf{v}_2 = [\mathbf{v}_1, \mathbf{v}_2] \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = [\mathbf{v}_1, \mathbf{v}_2]^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

*Example:*  $\mathbf{v}_1 = [1, 1]^t$ ,  $\mathbf{v}_2 = [-1, 1]^t$ , then

$$[\mathbf{v}_1, \mathbf{v}_2] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow [\mathbf{v}_1, \mathbf{v}_2]^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$$

## Gauss Transform

Define an elementary matrix as

$$E_{ik}(r) = I - r\mathbf{e}_i\mathbf{e}_k^t, \quad i > k \quad \Rightarrow \quad E_{ik}(r)^{-1} = I + r\mathbf{e}_i\mathbf{e}_k^t$$

A *Gauss* transform is a matrix of the form

$$\prod_{i=n}^{k+1} E_{ik} = E_{nk}E_{n-1,k} \cdots E_{k+1,k}$$

which can annihilate the components of a vector  $\mathbf{x}$  after index  $k$ .

*Examples*

$$G = E_{31}(-1)E_{21}(2) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \quad \Rightarrow \quad G\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

## Householder Transform (Elementary Reflector)

$$H = I - 2\mathbf{u}\mathbf{u}^t, \text{ where } \mathbf{u} \in R^n \text{ with } \|\mathbf{u}\|_2 = 1$$

$$H^t = H \text{ and } H^{-1} = H$$

Let  $\mathbf{x} = [3, 1, 5, 1]^t$ , then  $\|\mathbf{x}\|_2 = \sqrt{3^2 + 1^2 + 5^2 + 1^2} = 6$ .

Define  $\mathbf{v} = \mathbf{x} + \|\mathbf{x}\|_2\mathbf{e}_1$ , and let  $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\|_2$ , then

$$H = I - 2\mathbf{u}\mathbf{u}^t = \frac{1}{54} \begin{bmatrix} -27 & -9 & -45 & -9 \\ -9 & 53 & -5 & -1 \\ -45 & -5 & 29 & -5 \\ -9 & -1 & -5 & 53 \end{bmatrix}, \text{ and } H\mathbf{x} = \begin{bmatrix} -6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

## Jacobi Transform (Givens' Rotation)

$$J(i, k; \theta) = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & c & \cdot & -s & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & s & \cdot & c & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & 0 & \cdot & 0 & \cdot & 1 \end{bmatrix}$$

$$J_{hh} = 1 \text{ if } h \neq i \text{ or } h \neq k, \text{ where } i < k$$

$$J_{ii} = J_{kk} = c = \cos \theta$$

$$J_{ki} = s = \sin \theta, J_{ik} = -s = -\sin \theta$$

Let  $\mathbf{x}, \mathbf{y} \in R^n$ , then  $\mathbf{y} = J(i, k; \theta)\mathbf{x}$  implies that

$$y_i = cx_i - sx_k$$

$$y_k = sx_i + cx_k$$

$$c = \frac{x_i}{\sqrt{x_i^2 + x_k^2}}, \quad s = \frac{-x_k}{\sqrt{x_i^2 + x_k^2}},$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}, \quad \text{then } J(2, 4; \theta)\mathbf{x} = \begin{bmatrix} 1 \\ \sqrt{20} \\ 3 \\ 0 \end{bmatrix}$$

## Affine Transform with Applications

$$\mathbf{y} = A\mathbf{x} + \mathbf{t} \quad \Rightarrow \quad \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} e_i \\ f_i \end{bmatrix}$$

w	a	b	c	d	e	f	$ ad - bc $
1	0	0	0	0.16	0	0	0.01
2	0.85	0.04	-0.04	0.85	0	1.60	0.85
3	0.20	-0.26	0.23	0.22	0	1.60	0.07
4	-0.15	0.28	0.26	0.24	0	0.44	0.07

Table 1: An IFS consisting of 4 affine transforms for Fern

# Textures Generated by Fractal Models

Fractal models used to generate such textures as *ferns*, *Sierpinski triangles*, and *snowflakes* have recently received attention in many image compression field. Synthesis is based on the iterated function system (IFS) codes [1,2,3], which are nothing but a set of affine transformations. Let  $A \in R^{2 \times 2}$  and  $\mathbf{t} \in R^2$ . An affine transform on  $\mathbf{x} \in R^2$  is defined as  $A\mathbf{x} + \mathbf{t}$ . To describe the algorithm, we denote  $A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$ , with  $p_i = |a_i d_i - b_i c_i| \neq 0$ , and let  $\mathbf{t}_i$  denote  $\begin{bmatrix} e_i \\ f_i \end{bmatrix}$ . An algorithm based on IFS codes with  $K$  affine transforms is listed below. Experiments conducted using two sets of affine transformations to generate textures are given. The parameters of this fractal model are given in Tables 1 and 2, respectively. Two such synthesized textures are shown in the following Figure.

**Table 1. IFS codes for a fern**

i	$a_i$	$b_i$	$c_i$	$d_i$	$e_i$	$f_i$
1	0	0	0	0.16	0	0
2	0.85	0.04	-0.04	0.85	0	1.60
3	0.20	-0.26	0.23	0.22	0	1.60
4	-0.15	0.28	0.26	0.24	0	0.44

**Table 2. IFS codes for Sierpinski triangles**

i	$a_i$	$b_i$	$c_i$	$d_i$	$e_i$	$f_i$
1	0.5	0	0	0.5	0	0
2	0.5	0	0	0.5	1.0	0
3	0.5	0	0	0.5	0.5	0.5

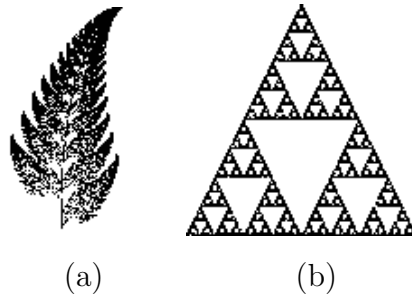


Figure 1: Textures synthesized using IFS codes.

### Fractal Generating Algorithm

- (1) Set  $m=0$  and randomly pick up an initial point  $\mathbf{x}^{(0)} \in R^2$ .
- (2) Select  $\{A_j, \mathbf{t}_j\}$  according to the probability distribution of  $\{r_j, 1 \leq j \leq K\}$ , where  $r_j = p_j / \sum_{i=1}^K p_i$  for  $1 \leq j \leq K$ .
- (3)  $\mathbf{x}^{(m+1)} \leftarrow A_j \mathbf{x}^{(m)} + \mathbf{t}_j$ .
- (4)  $m \leftarrow m + 1$ .
- (5) Repeat steps 2, 3, 4 until "convergence," for example,  $m = 1000$ , is achieved.
- (6) Plot  $\{\mathbf{x}^{(i)}\}$  for  $i = L$  to 1000, say  $L=100$ .

Convergence of this algorithm was studied by Barnsley [1], and Chu and Chen [3]. An IFS code consisting of two to five contractive affine transforms has been suggested [2,3] which generates self-similar images.

### References

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