

# Solutions for Test 2, ISA5305, Spring 2017

15:30-17:20, Monday, May 22, 2017

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(10%) 1(a) Let  $X$  be a continuous type of random variable that takes only nonnegative values. For any  $\beta > 0$ , prove that

$$P(X \geq \beta) \leq \frac{E(X)}{\beta}$$

**Proof:**

$$\begin{aligned} E(X) &= \int_0^{\infty} xf(x)dx = \int_0^{\beta} xf(x)dx + \int_{\beta}^{\infty} xf(x)dx \\ &\geq \int_{\beta}^{\infty} xf(x)dx \geq \int_{\beta}^{\infty} \beta f(x)dx, \\ &= \beta \int_{\beta}^{\infty} f(x)dx = \beta P(X \geq \beta) \end{aligned}$$

then

$$P(X \geq \beta) \leq \frac{E(X)}{\beta}$$

(10%) 1(b) Let  $X$  be a continuous type of random variable with mean  $E(X) = \mu$  and variance  $Var(X) = \sigma^2$ . For any  $\tau > 0$ , show that

$$P(|X - \mu| \geq \tau) \leq \frac{\sigma^2}{\tau^2}$$

**Proof:** Let  $A = \{x \mid |x - \mu| \geq \tau\}$ , then  $A^c = R - A$ .

$$\begin{aligned} Var(X) &= \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx = \int_A (x - \mu)^2 f(x)dx + \int_{A^c} (x - \mu)^2 f(x)dx \\ &\geq \int_A (x - \mu)^2 f(x)dx \geq \int_A \tau^2 f(x)dx, \\ &= \tau^2 \int_A f(x)dx = \tau^2 P(|X - \mu| \geq \tau) \end{aligned}$$

then

$$P(|X - \mu| \geq \tau) \leq \frac{Var(X)}{\tau^2} = \frac{\sigma^2}{\tau^2}$$

(5%) 1(c) Find the number of nonnegative integer solutions for

$$x_1 + x_2 + \cdots + x_k = n, \quad \text{where } n \geq k. \quad \mathbf{Ans} : C(n + k - 1, k - 1)$$

(5%) 1(d) Find the number of positive integer solutions for

$$x_1 + x_2 + \cdots + x_k = n, \quad \text{where } n \geq k. \quad \mathbf{Ans} : C(n - 1, k - 1)$$

(30%) 2. Fill the following blanks (no partial credits).

- (a) Let  $\{X_i, 1 \leq i \leq 9\}$  be a random sample of size 9 from  $N(2, 1)$ . Define  $\bar{X} = \frac{1}{9} \sum_{i=1}^9 X_i$  and  $Y = \sum_{i=1}^9 (X_i - 2)^2$ . Then

the moment-generating function  $M_{\bar{X}}(t) = \underline{e^{2t+(t^2/18)}}$

the moment-generating function  $M_Y(t) = \underline{\frac{1}{(1-2t)^{9/2}}}$

- (b) Let  $\{Z_i \sim N(0, 1), 1 \leq i \leq 10\}$  be a random sample of size 10. Define  $V = \sum_{i=1}^{10} Z_i$  and  $W = \sum_{i=1}^{10} Z_i^2$ . Then

the probability density function of  $V$ ,  $f_V(x) = \underline{\frac{1}{\sqrt{20\pi}} e^{-x^2/20}, -\infty < x < \infty}$

the moment-generating function  $M_W(t) = \underline{\frac{1}{(1-2t)^5}}$  and the p.d.f. of  $W$ ,  $f_W(x) = \underline{\frac{1}{\Gamma(5)2^5} x^4 e^{-x/2}}$

- (c) Let  $\{X_1, X_2, \dots, X_n\}$  be a random sample of size  $n$  from the *exponential distribution* with  $E(X_n) = 3$ . Define  $Y = \sum_{i=1}^n X_i$ . Then

the p.d.f. of  $Y$ ,  $f_Y(y) = \underline{\frac{1}{\Gamma(n)3^n} y^{n-1} e^{-y/3}, y > 0}$

the moment-generating function of  $Y$ ,  $\phi_Y(t) = \underline{\frac{1}{(1-3t)^n}}$

- (d) Let  $Z \sim N(0, 1)$ . Define  $Y = 3Z + 2$ , then

the p.d.f. of  $Y$ ,  $f_Y(y) = \underline{\frac{1}{\sqrt{18\pi}} \exp[-\frac{(y-2)^2}{18}], -\infty < y < \infty}$

the moment-generating function of  $Y$ ,  $\phi_Y(t) = \underline{e^{2t + \frac{9t^2}{2}}}$

- (e) Let  $\{X_i \sim \chi^2(1), 1 \leq i \leq n\}$  be a random sample and define  $W_n = (\sum_{i=1}^n X_i - n)/\sqrt{2n}$ . According to the Central Limit Theorem,

the limiting distribution function of  $W_n$   $\underline{\lim_{n \rightarrow \infty} P(W_n \leq w) = \int_{-\infty}^w \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \Phi(w)}$

the limiting moment-generating function is  $\underline{\lim_{n \rightarrow \infty} M_{W_n}(t) = e^{t^2/2}}$

(10%) 3. Let the r.v.  $X$  have the p.d.f.  $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$ ,  $0 < x < 1$ , where  $\alpha, \beta > 0$  are known positive integers.

Find  $E[X]$  and  $Var[X]$ .

**Hint:**  $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$  and  $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ .

**Proof:**

$$\begin{aligned} E[X] &= \int_0^1 x f(x) dx &&= \int_0^1 x \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} dx \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^\alpha (1-x)^{\beta-1} dx &&= \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \\ &= \frac{\alpha}{\alpha+\beta} \end{aligned}$$

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$$\begin{aligned} E[X^2] &= \int_0^1 x^2 f(x) dx &&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha+1} (1-x)^{\beta-1} dx \\ &= \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} &&= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} \end{aligned}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

(15%) 4. Let  $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$  be the order statistics of a random sample  $\{X_1, X_2, \dots, X_n\}$  from the exponential distribution with mean 2.

(a) Find the probability density function of  $X_{(1)}$ .

(b) Use the results of (a) to find  $E[X_{(1)}]$ .

(c) Find the probability density function of  $X_{(n)}$ .

(d) Use the result of (c) to find  $E[X_{(n)}]$ .

**Hint:** Let  $\{X_1, X_2, \dots, X_n\}$  be a random sample from  $U(0, 1)$ , then  $X_{(1)} = \min\{X_1, X_2, \dots, X_n\}$  and  $X_{(n)} = \max\{X_1, X_2, \dots, X_n\}$ .

**Proof:**

$$\begin{aligned} P(X_{(1)} \leq x) &= 1 - P(X_{(1)} > x) \\ &= 1 - \prod_{j=1}^n P(X_{(j)} > x) , \\ &= 1 - (1 - x)^n \end{aligned}$$

then,

$$(a) \quad f_{X_{(1)}}(x) = n(1 - x)^{n-1}, \quad 0 < x < 1$$

$$\begin{aligned} (b) \quad E(X_{(1)}) &= \int_0^1 xn(1 - x)^{n-1} dx \quad . \\ &= nB(2, n) = \frac{1}{n+1} \end{aligned}$$

$$\begin{aligned} P(X_{(n)} \leq x) &= \prod_{j=1}^n P(X_{(j)} \leq x) \\ &= \prod_{j=1}^n x \quad , \\ &= x^n \end{aligned}$$

then,

$$(c) \quad f_{X_{(n)}}(x) = nx^{n-1}, \quad 0 < x < 1$$

$$\begin{aligned} (d) \quad E(X_{(n)}) &= \int_0^1 nx^n dx \quad . \\ &= \frac{n}{n+1} \end{aligned}$$

(15%) 5. Let  $X$  have the p.d.f.  $f(x) = \beta x^{\beta-1}$ ,  $0 < x < 1$  for a given  $\beta > 0$ . Define  $Y = -3\beta \ln(X)$ .

- (a) Compute the distribution function of  $Y$ ,  $P(Y \leq y)$ , for  $0 < y < \infty$ .
- (b) Find the moment-generating function  $M_Y(t)$ .
- (c) Find  $E(Y)$  and  $Var(Y)$ .

**Proof:**

$$\begin{aligned}
 P(Y \leq y) &= P(-3\beta \ln(X) \leq y) \\
 &= P(X \geq e^{-y/3\beta}) \\
 &= \int_{e^{-y/3\beta}}^1 \beta x^{\beta-1} dx \\
 &= 1 - e^{-y/3}
 \end{aligned}$$

then, (a)  $f_Y(y) = \frac{1}{3}e^{-y/3}$   $0 < y < \infty$ ,  $Y$  is an exponential distribution with the parameter  $\theta = 3$ . (b)  $M_Y(t) = \frac{1}{1-3t}$ . (c)  $E(Y) = 3$  and  $Var(Y) = 9$ .