Solutions for Test 2: Probability Theory for ISA5305

Name:  
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(20pts) 1. For the following statements, mark a ○ if it is true, and mark a × otherwise.

(○) (a) The moment-generating function of the binomial distribution, with mean 40 and variance 8, is \((0.2 + 0.8e^t)^{50}\).

(○) (b) The p.m.f. of a geometric distribution with mean 2 is \(f(x) = \frac{1}{2}x^{-1}, x = 1, 2, 3, \ldots\).

(○) (c) The p.m.f. of a Poisson distribution with variance 2 is \(f(x) = \frac{e^{-2}2^x}{x!}, x = 0, 1, 2, \ldots\).

(○) (d) Let \(X\) have an exponential distribution with mean \(\theta > 0\), then for any \(a, b > 0\), we have \(P(X > a + b) = P(X > a) \times P(X > b)\).

(×) (e) Let \(X_1, X_2, X_3\) be mutually independent r.v.’s with Poisson distributions having means 1, 2, 4, respectively. Then \(\sum_{i=1}^{3} X_i\) has a Poisson distribution with variance 7.

(×) (f) Let \(X_i \sim N(1, 4), 1 \leq i \leq 4\), a random sample of size 4. Define \(Y = \sum_{i=1}^{4} X_i\). Then \(\frac{Y-4}{2} \sim N(0, 1)\).

(×) (g) Define \(\Gamma(x) = \int_{0}^{\infty} e^{-t}t^{x-1}dt\). Then \(\sum_{k=1}^{5} \Gamma(k) = 15\).

(×) (h) Let \(\{Z_i\}\) be a random sample of size \(n\) from the standard normal distribution. Then \(\overline{Z} = \frac{1}{n} \sum_{i=1}^{n} Z_i \sim N(0, \frac{1}{\sqrt{n}})\).

(×) (i) Let \(X \sim N(1, 2)\). Then \(M_X(t) = e^{t+2t^2}\).

(○) (j) Let \(X, Y \sim N(3, 2)\) be independent and \(W = X - Y\), then \(M_W(t) = e^{2t^2}\).
(30pts) 2. Choose the best (unique) solution for each of the following problems.

(2 ) (a) Let \( X \) have a density function \( f(x) = \frac{1}{2} e^{-x/2}, \ x \geq 0 \), then the variance of \( X \) is

1. 2, 2. 4, 3. \( \frac{1}{2} \), 4. \( \frac{1}{4} \), 5. none

(2 ) (b) Let \( X \) have a density function \( f(x) = \frac{1}{2} e^{-x/2}, \ x \geq 0 \), then the median of \( X \) is

1. \( e^{-1} \), 2. \( e^{-2} \), 3. \( 2ln2 \), 4. \( (ln2)/2 \), 5. none

(2 ) (c) Let \( X \) be a gamma distribution with p.d.f. \( f(x) = \frac{1}{16} x^2 e^{-x/2}, \ 0 \leq x \leq \infty \), then the variance of \( X \) is

1. 3, 2. 6, 3. 12, 4. 16, 5. none

(2 ) (d) Let \( X \) be a gamma distribution with p.d.f. \( f(x) = \frac{1}{16} x^2 e^{-x/2}, \ 0 \leq x \leq \infty \), then moment-generating function is

1. \( \frac{1}{(1-2t)^3} \), 2. \( \frac{1}{(1-3t)^2} \), 3. \( \frac{1}{1-2t} \), 4. \( \frac{1}{1-3t} \), 5. none

(2 ) (e) Let \( X \) be a \( \chi^2 \) distribution whose moment-generating function \( \phi(t) = \frac{1}{(1-2t)^2}, \ t < \frac{1}{2} \), then the variance \( \text{Var}(X) \) is

1. 1, 2. 2, 3. 4, 4. 8, 5. none

(2 ) (f) Let \( X \) be a \( \chi^2 \) distribution whose moment-generating function \( \phi(t) = \frac{1}{(1-2t)^2}, \ t < \frac{1}{2} \), then the first quartile of \( X \) is

1. \( 2 \ln(\frac{4}{3}) \), 2. \( 2 \ln(\frac{2}{3}) \), 3. 0.25, 4. 0.75, 5. none

(2 ) (g) Let \( X \sim \mathcal{N}(3, 4) \), then the moment-generating function of \( X \) is

1. \( e^{3t+4t^2} \), 2. \( e^{3t+2t^2} \), 3. \( \frac{1}{4-3e^t} \), 4. \( \frac{1}{3-4e^t} \), 5. none

(2 ) (h) Let \( X \sim \mathcal{N}(3, 4) \), then the median of \( X \) is

1. 1, 2. 2, 3. 3, 4. 4, 5. none

(2 ) (i) Let \( Z \sim \mathcal{N}(0, 1) \) and define \( X = 2Z + 3 \), then \( \text{Var}(X) \) equals

1. 1, 2. 2, 3. 3, 4. 4, 5. none

(2 ) (j) Let \( X \sim \mathcal{N}(3, 4) \) and define \( Y = (X - 3)/2 \), then \( \text{Var}(Y^2) \) equals

1. 1, 2. 2, 3. 3, 4. 4, 5. none
(30pts) 3. Fill the following blanks.

(a) \( f_X(x) = \left( \begin{array}{c} m \\ x \end{array} \right) p^x(1-p)^{m-x}, 0 \leq x \leq m \), \( f_W(w) = \left( \begin{array}{c} m+n \\ w \end{array} \right) p^w(1-p)^{m+n-w}, 0 \leq w \leq m+n \)

\( M_W(t) = (1-p+pe^t)^{m+n}, \ E(X) = mp, \ E(W) = (m+n)p \)

(b) \( f_X(x) = \frac{e^{-3x}x^3}{2!}, x = 0, 1, 2 \cdots \), \( f_Y(y) = \frac{e^{-3n(3n)y}}{y!}, y = 0, 1, 2 \cdots \)

\( M_Y(t) = e^{3n(e^t-1)} \), \( E(X_1) = 3 \), \( E(Y) = 3n \)

(c) \( f_{X_n}(x) = \frac{1}{2} e^{-x/2}, 0 < x < \infty \), \( f_Y(y) = \frac{1}{2n(n-1)!} y^{n-1} e^{-y/2}, y > 0 \)

\( M_Y(t) = \frac{1}{(1-2t)^n}, t < \frac{1}{2} \), \( E(X_n) = 2 \), \( E(Y) = 2n \)

(d) \( f_X(x) = \frac{1}{\sqrt{2\pi}a^2} e^{-(x-\mu)^2/2a^2}, -\infty < x < \infty \), \( f_Y(y) = \frac{1}{\sqrt{2\pi}a^2} e^{-(y-\beta)^2/2a^2}, -\infty < y < \infty \)

\( \phi_Y(t) = e^{\beta t + \frac{a^2 t^2}{2}}, \ E(Y) = \beta \), \( Var(Y) = \alpha^2 \)

(e) \( P(X = 4, Y = 4) = \left( \frac{1}{2} \right)^4 \times 5 \left( \frac{1}{2} \right)^5 = \frac{5}{512} \), \( P(X+Y = 8) = \frac{9}{512} \)

(f) Let the p.d.f. of \( X \) be \( f(x) = xe^{-x^2/2}, 0 < x < \infty \) and define \( Y = X^2 \). Then

\( E(Y) = 2 \), \( Var(Y) = 4 \), \( \phi_Y(t) = \frac{1}{1-2it} \) What is the distribution of \( Y \)?

Answer: Exponential distribution with mean 2
4. Given a random permutation of the integers in the set \{1, 2, 3, 4, 5\}, let \(X\) equal the number of integers that are in their natural position.

(a) Give the p.d.f. \(f_X(x)\), \(x = 0, 1, 2, 3, 4, 5\).

(b) Give the moment-generating function \(M_X(t)\).

(c) Find the mean \(E(X)\) and variance \(Var(X)\).

Solution:

(a) \(f(0) = \frac{44}{120}, f(1) = \frac{45}{120}, f(2) = \frac{20}{120}, f(3) = \frac{10}{120}, f(4) = 0, f(5) = \frac{1}{120}\).

(b) \(M(t) = \frac{44}{120} + \frac{45}{120}e^t + \frac{20}{120}e^{2t} + \frac{10}{120}e^{3t} + \frac{1}{120}e^{5t}\)

(c) \(M'(0) = \frac{45}{120} + \frac{40}{120} + \frac{30}{120} + \frac{5}{120} = \frac{120}{120} = 1\)

\(M''(0) = \frac{45}{120} + \frac{80}{120} + \frac{90}{120} + \frac{25}{120} = \frac{240}{120} = 2\)

Thus, \(E(X) = 1\) and \(Var(X) = M''(0) - [M'(0)]^2 = 1\).