(22 pts) 1. Mark ○ if the statement is true, and mark × otherwise, or give your comments.

( × ) (a) Every overdetermined linear system must not have a unique solution.
( × ) (b) Every nonsingular matrix has an LU-decomposition.
( ○ ) (c) If \( \lambda \) is an eigenvalue of matrix \( A \), then \( \lambda - \mu \) must be an eigenvalue of \( A - \mu I \).
( ○ ) (d) \( L: \mathbb{R}^m \rightarrow \mathbb{R}^n \) is a linear transform, then \( \text{Ker}(L) \) is a vector subspace of \( \mathbb{R}^m \).
( × ) (e) Let \( X, Y \) be 1-dimensional vector subspaces of \( \mathbb{R}^2 \) and \( X \perp Y \), then \( \mathbb{R}^2 = X \cup Y \).
( ○ ) (f) The product of eigenvalues of a square matrix \( A \) equals the determinant of \( A \).
( ○ ) (g) All eigenvalues of a real symmetric matrix must be real.
( × ) (h) Every nonsingular square matrix can be diagonalized.
( × ) (i) Let \( A, B \in \mathbb{R}^{n \times n} \) be symmetric, then \( AB = BA \).
( ○ ) (j) Similar matrices always have the same eigenvalues.
( ○ ) (k) Let \( x \in \mathbb{R}^n \) with \( \|x\|_2 = 3 \). If \( A \in \mathbb{R}^{n \times n} \) is orthogonal, then \( \|Ax\|_2 = 3 \).
(28 pts) 2. Answering each of the following questions.

(a) Let $w, x, y, z \in \mathbb{R}^n$ be orthonormal vectors, then $\|w + x + 3y - 5z\|_2 = 6$

(b) Let $H_1, H_2, \ldots, H_m \in \mathbb{R}^{k \times k}$ be Householder matrices. Then $\det(\prod_{j=1}^{m} H_j) = (-1)^m$

(c) Let $A \in \mathbb{R}^{n \times n}$ have eigenvalues $2, 4, 6, \ldots, 2n$. The trace of $A = n(n + 1)$

(d) Let $A \in \mathbb{R}^{3 \times 3}$ have eigenvalues $\lambda(A) = \{5, 3, 2\}$. Then $\lambda((A - I)^{-1}) = \{0.25, 0.5, 1\}$

(e) Let $V = \text{Span}(e_n)$ be a subspace of $\mathbb{R}^n$, then $\dim(V^\perp) = n - 1$

(f) Let $x = [2, 0, -2]^t$, $y = [0, 2, -2]^t$, then the angle between $x$ and $y$, $\angle(x, y) = \frac{\pi}{3}$ or $60^\circ$

(g) Let $A \in \mathbb{R}^{3 \times 3}$ have eigenvalues $5, 3, 1$. Then $\det(A - 2I) = -3$
(20 pts) 3. Let \( A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \).

(a) Find the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of matrix \( A \) and their corresponding unit eigenvectors \( u_1 \) and \( u_2 \).

(b) Find the trace of \( A \) and the determinant of \( A \).

(c) Let \( U = [u_1, u_2] \), compute \( U^tAU \).

(d) Find the eigenvalues \( \mu_1 \) and \( \mu_2 \) of matrix \( A^{-1} \).

(e) Find the trace of \( A^{-1} \) and the determinant of \( A^{-1} \).

(f) Do the same problems of (a \( \sim \) e) for matrix \( B \).

\textbf{Ans:} \( p(A) = (\lambda + 2)(\lambda + 4) = 0 \).

\( a) \lambda_1 = -2, \ u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), and \( \lambda_2 = -4, \ u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \), and

\( b) \ tr(A) = \lambda_1 + \lambda_2 = -6 \) and \( det(A) = \lambda_1 \times \lambda_2 = 8 \).

\( c) U = [u_1, u_2], \ U^{-1}AU = U^tAU = diag(-2, -4) \).

\( d) \ \mu_1 = -\frac{1}{2} \) and \( \mu_2 = -\frac{1}{4} \) for \( A^{-1} \).

\( e) \ tr(A^{-1}) = -\frac{3}{4} \) and \( det(A^{-1}) = \frac{1}{8} \).

\textbf{B-Ans:} \( p(B) = (\lambda - 1)^2 = 0 \).

\( Ba) \ \lambda_1 = \lambda_2 = 1, \ u_1 = u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).

\( Bb) \ tr(B) = \lambda_1 + \lambda_2 = 2 \) and \( det(B) = \lambda_1 \times \lambda_2 = 1 \).

\( Bc) U = [u_1, u_2], \ U^tBU = [1, 1; 1, 1] \) and

\( Bd) \ \mu_1 = 1 \) and \( \mu_2 = 1 \) for \( B^{-1} \).

\( Be) \ tr(B^{-1}) = 2 \) and \( det(B^{-1}) = 1 \).
(20 pts) 4. Let $A$ be a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and corresponding orthonormal eigenvectors $u_1, u_2, \ldots, u_n$. For each nonzero vector $x \in \mathbb{R}^n$, the Rayleigh quotient $\rho(x)$ is defined by

$$\rho(x) = \frac{\langle Ax, x \rangle}{\langle x, x \rangle}$$

(a) For $x = \sum_{i=1}^{n} c_i u_i$ with $\sum_{i=1}^{n} c_i^2 = 1$, prove that $\rho(x) = \sum_{i=1}^{n} \lambda_i c_i^2$

(b) Show that $\lambda_n \leq \rho(x) \leq \lambda_1$

(c) Show that for $x \neq 0$, $\text{Min}\{\rho(x)\} = \lambda_n$ and $\text{Max}\{\rho(x)\} = \lambda_1$

(a) Proof:

$$\langle x, x \rangle = x^t x = \left( \sum_{i=1}^{n} c_i u_i \right)^t \left( \sum_{j=1}^{n} c_j u_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j u_i^t u_j = \sum_{i=1}^{n} c_i^2 = 1$$

$$\langle Ax, x \rangle = x^t Ax = \left( \sum_{i=1}^{n} c_i u_i \right)^t \left( \sum_{j=1}^{n} \lambda_j c_j u_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_j c_i c_j u_i^t u_j = \sum_{i=1}^{n} \lambda_i c_i^2$$

Then

$$\rho(x) = \frac{\langle Ax, x \rangle}{\langle x, x \rangle} = \sum_{i=1}^{n} \lambda_i c_i^2$$

(b) Proof:

$$\lambda_n \leq \lambda_i \leq \lambda_1, \quad \forall \ 1 \leq i \leq n, \text{ then}$$

$$\lambda_n = \sum_{i=1}^{n} \lambda_n c_i^2 \leq \sum_{i=1}^{n} \lambda_i c_i^2 \leq \sum_{i=1}^{n} \lambda_1 c_i^2 = \lambda_1$$

(c) Proof: In (a), let $c_1 = 1$ and $c_j = 0$, $2 \leq j \leq n$, then $\rho(x) = \lambda_1$. By (b), $\text{Max}\{\rho(x)\} = \lambda_1$. Similarly, $\text{Min}\{\rho(x)\} = \lambda_n$
(10 pts) 5. Let \( b = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix} \), \( C = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \).

Give a single Matlab command to solve each of the following questions for \( \text{a} \sim \text{h} \) and answer the questions for \( \text{i} \sim \text{h} \).

(a) Randomly generate a 3 by 3 matrix \( A \) whose elements are integers in \([0, 10)\). \( A = \text{fix}(10*\text{random('unif',0,1,3,3)}) \)

(b) Input vector \( b \).
\( b = [1; -5; 4] \)

(c) Solve the linear system \( Ax = b \) for \( x \).
\( x = A\backslash b \)

(d) Input matrix \( C \) given above.
\( C = [-2,1,0; 1,-2,0; 0,0,2] \)

(e) Compute the characteristic polynomial for \( C \). \( p=\text{poly}(C) \)

(f) Compute the eigenvalues and eigenvectors of \( C \). \( [U, D]=\text{eig}(C) \)

(g) Compute the inverse of matrix \( C \). \( \text{inv}(C) \)

(h) Compute the rank of matrix \( C \). \( \text{rank}(C) \)

(i) Compute the \( LU - \) decomposition of the matrix \( C \). \( [L,U,P]=\text{lu}(C) \)

(j) Compute the \( QR - \) factorization of the matrix \( C \). \( [Q, R]=\text{qr}(C) \)