Homework 1: Solving Ax=b, Determinants, and Orthogomality

This exercise requires you to write an *efficient* and *effective* Matlab, C/C++, or Java program to solve a linear system of equations $A\mathbf{x} = \mathbf{b}$ by the strategy of LU decomposition with *partial pivoting*.

(1) Apply the Gaussian Elimination with Partial Pivoting method to solve the following equations.

$$\begin{aligned} 4.00001x &+ 1.00000y + 2.00000z = 4.00001 \\ 10.00000x &- 0.10000y + 3.00000z = 12.80000 \\ 5.00000x &+ 3.00000y + 1.00000z = 12.00000 \end{aligned}$$

(a) Write this equation as $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(b) Find $PA = LU$, where L is unit lower$-\Delta$ and U is upper$-\Delta$ (using either Matlab or C/C++/Java programs).

(c) Apply Gaussian elimination and back substitution to solve $PA\mathbf{x} = P\mathbf{b}$.

(d) Can you get the same solution without using the partial pivoting strategy for this problem?

(2) The $n \times n$ Hilbert matrix $H_n = [h_{ij}]$ is defined by

$$h_{ij} = \frac{1}{i+j-1}, \quad i, j, = 1, 2, \cdots, n$$

H_n can be generated using the MATLAB function $\mathbf{hilb}(n)$. It is well known that the Hilbert matrix is nortoriously ill-conditioned.

(a) Generate H_n, and $\mathbf{x}_n = \mathbf{1}_n = \mathbf{ones}(n, 1)$ for $n = 8, 12, 16, 20, 24$. In each case, construct $H_n\mathbf{x}_n = \mathbf{b}_n$ so that $\mathbf{1}_n$ is the exact solution. Then, by the given H_n and $\mathbf{b}_n$, applying the LU decomposition with partial pivoting strategy to solve $\mathbf{x}_n$.

(b) Find the determinant of each H_n in (**a**).

(c) Compute the condition number with p-norm, p=1,2,∞ for each H_n in (**a**).

(d) Discuss your solutions.

(3) Let $A = [a_{ij}] \in R^{n\times n}$ be defined as $a_{ii} = 2$, $a_{ij} = -1\ if\ |i-j| = 1$, and 0 otherwise.

(a) Write down A_2, A_3 in their matrix forms, what are $|A_2|$ and $|A_3|$?

(b) Show that $|A_{n+1}| = 2|A_n| - |A_{n-1}|$ and compute $|A_n|$ in terms of n.

(c) Write Matlab codes to generate a matrix A_n and its determinant for each given n.

(4) To further understand the properties of Householder matrices such as symmetry, orthogonality, and the determinant, implement the following steps in Matlab environments.

```
for n=3:6
  v=rand(n,1);
  u=v/norm(v,2);
  H=eye(n)-2*u*u';
  [norm(H'-H,1), det(H), norm(H'*H-eye(n),1)]
  x=ones(n,1);
  e1=zeros(n,1);  e1(1,1)=1;
  v=x-norm(x,2)*e1;
  u=v/norm(v,2);
  H=eye(n)-2*u*u';
  y=H*x
end
```