Test 1: Linear Algebra for ISA5305

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(22 pts) 1. Mark ○ if the statement is true, and mark × otherwise, or give your comments.

( ) (a) Not every underdetermined linear system has a solution.

( ) (b) Not every nonsingular matrix has an LU-decomposition.

( ) (c) If λ is an eigenvalue of matrix A, then λ^m must be an eigenvalue of A^m.

( ) (d) L : R^m → R^n is a linear transform, then Ker(L) is a vector subspace of R^m.

( ) (e) Let X, Y be 1-dimensional vector subspaces of R^2 and X ⊥ Y, then R^2 = X ⊕ Y.

( ) (f) The product of eigenvalues of A equals the product of diagonal elements of A.

( ) (g) All eigenvalues of a real symmetric matrix must be distinct.

( ) (h) Every nonsingular square matrix can be diagonalized.

( ) (i) Let A, B ∈ R^{n×n} be symmetric, then (A + B)(A - B) = A^2 - B^2.

( ) (j) Similar matrices always have the same eigenvalues.

( ) (k) Let x ∈ R^n with ∥x∥_2 = 5. If A ∈ R^{n×n} is orthogonal, then ∥Ax∥_2 = 25.
(28 pts) 2. Answering each of the following questions.

(a) Let $u, v, w \in \mathbb{R}^n$ be orthonormal vectors, then $\|u - 2v + 2w\|_2 = ?$

(b) Let $H_1, H_2, \cdots, H_k \in \mathbb{R}^{n \times n}$ be Householder matrices. Then $\det(\prod_{j=1}^{k} H_j) = ?$

(c) Let $A \in \mathbb{R}^{n \times n}$ have eigenvalues $1, 3, 5, \cdots, 2n - 1$. What is the trace of $A$?

(d) Let $A \in \mathbb{R}^{3 \times 3}$ have $\lambda(A) = \{1, 2, 5\}$. What is $\lambda(A^{-1})$?

(e) Let $V = \text{Span}(e_1, e_3) \subseteq \mathbb{R}^n$, what is $\dim(V^\perp)$?

(f) Let $x = [2, 0, -2]^t$, $y = [0, 2, -2]^t$, then the angle between $x$ and $y$, what is $\angle(x, y)$?

(g) Let $A \in 3 \times 3$ have eigenvalues $3, 4, 6$ what are the eigenvalues of $(A - 2I)^{-1}$?
(20 pts) 3. Let \( A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \).

(a) Find the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of matrix \( A \) and their corresponding unit eigenvectors \( u_1 \) and \( u_2 \).

(b) Find the trace of \( A \) and the determinant of \( A \).

(c) Let \( U = [u_1, u_2] \), compute \( U^tAU \).

(d) Find the eigenvalues \( \mu_1 \) and \( \mu_2 \) of matrix \( A^{-1} \).

(e) Find the trace of \( A^{-1} \) and the determinant of \( A^{-1} \).
(20 pts) 4. Let \( A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \), then \( A^2 = \begin{bmatrix} 5 & -4 & 0 \\ -4 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} \), and define

\[ \alpha = \min_{\|x\|_2 = 1} \{ x^t A^5 x \}, \quad \beta = \max_{\|y\|_2 = 1} \{ y^t A^5 y \}. \]

(a) Find the eigenvalues \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) of matrix \( A \) and their corresponding unit eigenvectors \( u_1, u_2, \) and \( u_3. \)

(b) Find the eigenvalues \( \mu_1, \mu_2, \) and \( \mu_3 \) of matrix \( A^2 \) and their corresponding unit eigenvectors \( v_1, v_2, \) and \( v_3. \)

(c) Find the eigenvalues \( \tau_1, \tau_2, \) and \( \tau_3 \) of matrix \( A^5 \) and their corresponding unit eigenvectors \( w_1, w_2, \) and \( w_3. \)

(d) Compute the values of \( \alpha \) and \( \beta. \)
Let $b = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Give a single Matlab command to solve each of the following questions for $a \sim j$.

(a) Randomly generate a 3 by 3 matrix $A$ whose elements are integers in $[0, 10)$.
(b) Input vector $b$.
(c) Solve the linear system $Ax = b$ for $x$.
(d) Input matrix $C$ given above.
(e) Compute the characteristic polynomial for $C$.
(f) Compute the eigenvalues and eigenvectors of $C$.
(g) Compute the trace of matrix $C$.
(h) Compute the rank of matrix $C$.
(i) Compute the $LU$ – decomposition of the matrix $C$.
(j) Compute the $QR$ – factorization of the matrix $C$. 