Covariance and Correlation Coefficient

For artibrary random variables X and Y, and constants a and b, we have

$$E[aX + bY] = aE[X] + bE[Y]$$

Proof: We'll show for the continuous case, the discrete case can be similarly proved.

$$\begin{aligned}
E[aX + bY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(ax + by)f(x,y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}axf(x,y)dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}byf(x,y)dxdy \\
&= \int_{-\infty}^{\infty}ax\left[\int_{-\infty}^{\infty}f(x,y)dy\right]dx + \int_{-\infty}^{\infty}by\left[\int_{-\infty}^{\infty}f(x,y)dx\right]dy \\
&= a\int_{-\infty}^{\infty}xf_X(x)dx + b\int_{-\infty}^{\infty}yf_Y(y)dy \\
&= aE[X] + bE[Y]
\end{aligned}$$

Similarly,

$$E\left(\sum_{i=1}^{n}a_iX_i\right) = \sum_{i=1}^{n}a_iE(X_i)$$

Furthermore,

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xyf(x,y)dxdy$$

[*Example*] Let $f(x,y) = \frac{1}{3}(x+y),\ \ 0 < x < 1,\ 0 < y < 2$, and $f(x,y) = 0$ elsewhere.

$$E[XY] = \int_0^1\int_0^2 xyf(x,y)dydx = \int_0^1\int_0^2 xy\frac{1}{3}(x+y)dydx = \frac{2}{3}$$

- Let X and Y be independent random variables, then

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xyf_X(x)f_Y(y)dxdy = \left[\int_{-\infty}^{\infty}xf_X(x)dx\right]\cdot\left[\int_{-\infty}^{\infty}yf_Y(y)dy\right] = E(X)\cdot E(Y)$$

- The covariance between r.v.'s X and Y is defined as

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)f(x,y)dydx = E(XY) - \mu_X\mu_Y$$

- If X and Y are independent r.v.s, then $Cov(X,Y) = 0$.
- The correlation coefficient is defined by $\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$

Expectation and Covariance Matrix

Let $X_1, X_2, \ldots, X_n$ be random variables such that the expectation, variance, and covariance are defined as follows.

$$\mu_j = E(X_j), \quad \sigma_j^2 = Var(X_j) = E[(X_j - \mu_j)^2]$$

$$Cov(X_i, X_j) = E[(X_i - \mu_i)(X_j - \mu_j)] = \rho_{ij}\sigma_i\sigma_j$$

Suppose that $\mathbf{X} = [X_1, X_2, \ldots, X_n]^t$ is a random vector, then the expected mean vector and covariance matrix of $\mathbf{X}$ is defined as

$$\begin{aligned} E(\mathbf{X}) &= [\mu_1, \mu_2, \ldots, \mu_n]^t = \mu \\ Cov(\mathbf{X}) &= E[(\mathbf{X} - \mu)(\mathbf{X} - \mu)^t] \\ &= [E((X_i - \mu_i)(X_j - \mu_j))] \end{aligned}$$

Theorem 1: Let $X_1, X_2, \ldots, X_n$ be n independent r.v.'s with respective means $\{\mu_i\}$ and variances $\{\sigma_i^2\}$, then $Y = \sum_{i=1}^n a_i X_i$ has mean $\mu_Y = \sum_{i=1}^n a_i \mu_i$ and variance $\sigma_Y^2 = \sum_{i=1}^n a_i^2 \sigma_i^2$, respectively.

Theorem 2: Let $X_1, X_2, \ldots, X_n$ be n independent r.v.'s with respective moment-generating functions $\{M_i(t)\}$, $1 \le i \le n$, then the moment-generating function of $Y = \sum_{i=1}^n a_i X_i$ is $M_Y(t) = \prod_{i=1}^n M_i(a_i t)$.

Multivariate (Normal) Distributions

♢ (Gaussian) Normal Distribution: $X \sim N(u, \sigma^2)$

$$f_X(x) = f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp^{-(x-u)^2/2\sigma^2} \quad for -\infty < x < \infty$$

$$mean \;\; and \;\; variance: \;\; E(X) = u, \;\; Var(X) = \sigma^2$$

♢ (Gaussian) Normal Distribution: $X \sim N(\mathbf{u}, C)$

$$f_X(\mathbf{x}) = f(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}[det(C)]^{1/2}} e^{-(\mathbf{x}-\mathbf{u})^t C^{-1}(\mathbf{x}-\mathbf{u})/2} \quad for \;\; \mathbf{x} \in R^d$$

$$mean\ vector \;\; and \;\; covariance\ matrix: \;\; E(X) = \mathbf{u}, \;\; Cov(X) = C$$

♢ Simulate $\mathbf{X} \sim N(\mathbf{u},\ C)$

(1) $C = LL^t$, where L is lower-Δ.

(2) Generate $\mathbf{y} \sim N(\mathbf{0}, I)$.

(3) $\mathbf{x} = \mathbf{u} + L * \mathbf{y}$

(4) Repeat Steps $(\mathbf{2}) and (\mathbf{3})$ M times.

```
%  Simulate N([1 3]', [4,2; 2,5])
%
n=30;
X1=random('normal',0,1,n,1);
X2=random('normal',0,1,n,1);
Y=[ones(n,1), 3*ones(n,1)]+[X1,X2]*[2 1; 0, 2];
Yhat=mean(Y)  % estimated mean vector
Chat=cov(Y)   % estimated covariance matrix
% Z=[X1, X2];
```

Plot a 2D standard Gaussian Distribution

```
x=-3.6:0.3:3.6;
y=x';
X=ones(length(y),1)*x;
Y=y*ones(1,length(x));
Z=exp(-(X.^2+Y.^2)/2+eps)/(2*pi);
mesh(Z);
title('f(x,y)= (1/2\pi)*exp[-(x^2+y^2)/2.0]')
```