(1) Consider equiprobable one-dimensional problems of dichotomy with samples distributed according to the Rayleigh probability density function in each class as given below.

\[ p(x|\omega_i) = \frac{x}{\sigma_i^2}e^{\left(-\frac{x^2}{2\sigma_i^2}\right)} \text{ for } x \geq 0 \]

Consider two classes with \( \sigma_1 = 1 \) and \( \sigma_2 = 3 \).

(a) Draw the density functions for these classes.
(b) Compute the Bayes decision boundary and plot it.
(c) Compute the Bayes error rate.

(2) This problem is to learn how far the Bayes decision theory is from a practical application by simulations. Let \( X|\omega_1 \sim N(\mathbf{u}_1,C) \) and \( X|\omega_2 \sim N(\mathbf{u}_2,C) \), where

\[ \mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \end{bmatrix} \]

Assume the equal priors. Now do the following tasks.

(a) Find an optimal decision rule and its error rate.
(b) Generate 100 patterns \( \{x_i\} \) independently from \( N(\mathbf{u}_1,C) \).
(c) Generate 100 patterns \( \{y_i\} \) independently from \( N(\mathbf{u}_2,C) \).
(d) Apply MLE to find \( \mathbf{u}_1 \) and \( \hat{C}_1 \).
(e) Apply MLE to find \( \hat{\mathbf{u}}_2 \) and \( \hat{C}_2 \).
(f) Compute the error rate based on MAP (a quadratic classifier).
(g) How close is the result in (f) to that in (a)?

(3) This problem is to verify the density estimation methods by the Parzen window and the K-nn distance approaches.

(a) Generate 400 patterns independently from \( N(0,1) \).
(b) Compute the estimated probability density \( \hat{p}(x) \) using the density function of the standard normal distribution as a kernel with some appropriate values of \( h \) and \( \alpha \) introduced in class (report your \( h \) and \( \alpha \) values).

(c) Compute the estimated probability density \( \hat{f}(x) \) using the K-nn distance approach with an appropriate value \( K \).

(d) Compare the results of (b) and (c) with the true density by plotting \( \hat{p}(x) \) and \( \hat{f}(x) \) vs. \( x \), respectively.

(e) Repeat steps (a) \sim (d) for the exponential distribution \( f(x) = \frac{1}{2}e^{-x/2}, x \geq 0 \).

(4) Show that for the lognormal distribution

\[
p(x) = \frac{1}{\sqrt{2\pi\sigma x}} \exp\left(-\frac{[ln x - \theta]^2}{2\sigma^2}\right), \quad x > 0
\]

the maximum likelihood (ML) estimates of \( \theta \) and \( \sigma^2 \) for a sample of size \( N \) are given by

\[
\hat{\theta}_{ML} = \frac{1}{N} \sum_{k=1}^{N} \ln(x_k), \quad \hat{\sigma}_{ML}^2 = \frac{1}{N} \sum_{k=1}^{N} (\ln(x_k) - \hat{\theta}_{ML})^2
\]