Exam I for CS3332

October 26, 2001

(5) A01. The final scores of *59* students taking a certain course in Fall, 1999 are listed below.

61	72	77	58	67	70	76	70	76	83
42	58	49	74	65	55	90	80	31	61
53	82	90	51	48	55	84	70	48	76
61	76	70	70	66	50	80	73	77	43
71	99	66	63	63	52	54	80	67	29
52	83	62	60	61	86	61	70	73	

(a) Find the order statistics of these data.

(b) Find the first, second, and third quartiles.

(c) Give a five-number summary.

(d) Draw a box-and-whisker diagram.

(e) Are there any outliers? Explain.

(5) A02. The following data give the math and verbal scores of a certain test for 10 students.

(0,1) (1,0) (1,1) (2,1) (2,6)
(3,1) (3,3) (3,10) (5,9) (6,7)

Find the best-fitting (least squares regression) line.

(5) A03. Show that (a) $\sum_{k=0}^{n}(-1)^k \binom{n}{k} = 0$, (b) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

(5) A04. Let an experiment be drawing five cards at random without replacement from a deck of 52 poker cards. The sample size is then $C(52, 5) = 2,598,960$. Find the size of the following events.

(a) Four of a kind (four cards of equal face vaule and one card of a different value).

(b) Full house (one pair and one triple of cards with equal face value).

(c) Three of a kind (three equal face values plus two cards of different values).

(d) Two pairs (two pairs of equal face value plus one card of a different value).

(e) One pair (one pair of equal face value plus three cards of different values).

(5) A05. In a certain random experiment, let A and B be two events such that $P(A) = 0.7$, $P(B) = 0.5$, and $P((A \cup B)') = 0.1$.

(a) $P(A \cap B) =$ ______

(b) $P(A|B) =$ ______

(c) $P(B|A) =$ ______

(d) Are A and B independent events? Explain.

(5) A06. In a certain random experiment, let C and D be two events such that $P(C) = 0.8$, $P(D) = 0.4$, and $P((C \cup D)') = 0.12$.

(a) $P(C \cap D) =$ 0.32

(b) $P(D|C) =$ 0.40

(c) $P(C|D) =$ 0.80

(d) Are C and D independent events? Explain.

(*) Yes, since $P(C \cap D) = 0.32 = 0.8 \times 0.4 = P(C)P(D)$.

(5) A07. A box contains four marbles numbered 1 through 4. The marbles are selected one at a time without replacement. A match occurs if marble numbered k is the kth marble selected. Let the event A_i, denote a match on the ith draw, i=1,2,3,4.

(a) Find P(A_i), for i=1,2,3,4.

(b) Find P($A_i \cap A_j$), where $1 \leq i < j \leq 4$.

(c) Find P($A_i \cap A_j \cap A_k$), where $1 \leq i < j < k \leq 4$.

(d) Find P($A_1 \cup A_2 \cup A_3 \cup A_4$).

(5) A08. Let the r.v. X have the p.d.f. $f(x) = (|x| + 1)^2/9$, $x = -1, 0, 1$. Compute $E(X)$, $E(X^2)$, and $E(3X^2 - 2X + 4)$.

(5) A09. Given $E(X + 4) = 10$ and $E((X + 4)^2) = 116$, determine $Var(X + 4)$, $E(X)$, and $E[(X - E(X))^2]$.

(5) A10. Suppose that 2000 points are independently and randomly selected from the unit square $S = \{(x, y) : 0 \leq x, y \leq 1\}$. Let Y equal the number of points that fall in $R = \{(x, y) : |x + y| \leq 1 \textit{ and } |x - y| \leq 1\}$.

(a) How is Y distributed?

(b) Give the mean and variance of Y.

(c) What is the expected value of $Y/500$?

(d) What is P($Y \leq 100$)?

(5) A11. Suppose that 2000 points are independently and randomly selected from the unit square $S = \{(x, y) : 0 \le x, y \le 1\}$. Let Y equal the number of points that fall in $A = \{(x, y) : x^2 + y^2 \le 1\}$.

(a) How is Y distributed?

(b) Give the mean and variance of Y.

(c) What is the expected value of $Y/500$?

(d) What is $\mathrm{P}(Y \le 100)$?

(5) A12. It is believed that 20% of Americans do not have any health insurance. Let X equal the number with no health insurance in a random sample of $n = 15$ Americans.

(a) How is X distributed?

(b) Find the mean and variance of Y.

(c) $\mathrm{P}(X \ge 2)$.

(5) A13. Consider a random experiment of casting a pair of unbiased six-sided dice and let the r.v. X equal the *smaller* of the outcomes if they are different and the common value if they are equal.

(a) Find the p.d.f. of r.v. X.

(b) Draw a probability histogram.

(c) Find the expectation and variance of r.v. X.

(5) A14. Consider a random experiment of casting a pair of unbiased six-sided dice and let the r.v. Y equal the *larger* of the outcomes if they are different and the common value if they are equal.

(a) Find the p.d.f. of r.v. Y.

(b) Draw a probability histogram.

(c) Find the expectation and variance of r.v. Y.

(5) A15. In a lottery, a 3-digit integer is selected at random from 000 to 999, inclusive. Let X be the integer selected on a particular day.

(a) Find the p.d.f. of the r.v. X.

(b) Find the mean of the r.v. X.

(c) Find variance of the r.v. X.

(5) A16. For the following distributions of X, find $\mu = E(X)$ and $\sigma^2 = E((X - \mu)^2)$.

(a) $f(x) = \gamma^x e^{-\gamma}/x!, x = 0, 1, 2, \ldots$, where $\gamma > 0$.

(b) $f(x) = (1-p)^{x-1}p, \ x = 1, 2, \ldots, \ where\ 0 < p < 1.$

(5) A17. Let the r.v. X have a Poisson discribution with the p.d.f. $f(x) = \lambda^x e^{-\lambda}/x!, \ x = 0, 1, 2, \ldots, \infty$, where $\lambda > 0$ is a known parameter.

(a) Find the mean, $E(X)$.

(b) Find the variance, Var(X).

(c) Find the mode of the probability density function f.

Solutions for Exam 1, 2001

A01(a) See class notes.

A01(b) $q_1 = 55$, $q_3 = 76$, the median = *67*.

A01(c) See class notes.

A02 $y = \frac{213}{152}x + \frac{39}{152} = 1.4013x + 0.2565$.

A11(a) $b(2000, \frac{\pi}{4})$, i.e., binomial distribution

A11(b) 500π, $500\pi(1$-$\frac{\pi}{4})$

A11(c) π

A11(d) $\sum_{k=0}^{100} C(2000, k)(\pi/4)^k(1-\pi/4)^{2000-k}$

A07(a) $\mathrm{P}(A_i) = 3!/4!$, for i=1,2,3,4.

A07(b) $\mathrm{P}(A_i \cap A_j) = 2!/4!$, where $1 \leq i < j \leq 4$.

A07(c) $\mathrm{P}(A_i \cap A_j \cap A_k) = 1!/4!$, where $1 \leq i < j < k \leq 4$.

A07(d) $\mathrm{P}(A_1 \cup A_2 \cup A_3 \cup A_4) = 1 - 1/2! + 1/3! - 1/4! = 5/8$

A13(a) $f(x) = \frac{13-2x}{36}$, for $x = 1, 2, 3, 4, 5, 6$

A13(b) Easy bar graph.

A13(c) $E[X]$=91/36, $\sigma^2 = 2555/1296 \approx 1.971$

A15(a) $f(x) = 1/1000,\ 000 \leq x \leq 999$

A15(b) $E[X] = 999/2 = 499.5$

A15(c) $Var(X) = (10^6 - 1)/12 = 333333/4 = 83333.25$

A16(a) $E[X] = \gamma$, $Var(X) = \gamma$

A16(b) $E[X] = 1/p$, $Var(X) = (1-p)/p^2$

A17(a) $E[X] = \lambda$

A17(b) $Var[X] = \lambda$

A17(c) $themodeis\lfloor\lambda\rfloor$

Exam II for CS3332

November 30, 2001

(5) B01. Let X have a Poisson distribution of variance 4. Find (a) $\mathrm{P}(X = 4)$, (b) $\mathrm{P}(2 \leq X \leq 6)$.

(5) B02. Derive the moment-generating functions of (a) Geometric, (b) Binomial, (c) Poisson distributions, respectively. Then find the means and variances of the above distributions by means of *moments*.

(5) B03. Let a r.v. X have the probability density function $f(x) = \frac{1}{2}sin(x)$, $0 \leq x \leq \pi$.

(a) Find the mean μ and variance σ^2.

(b) Sketch the graph of the p.d.f. of X.

(c) Sketch the graph of the distribution function of X.

(5) B04. Let $f(x) = (x + 1)/2$, $-1 < x < 1$. Find (a) $q_1 = \pi_{0.25}$, (b) $q_2 = \pi_{0.50}$, (c) $q_3 = \pi_{0.75}$.

(5) B05. Write down the probability density function and the space of the random variable for each of the following distributions.

(a) A binomial distribution with mean 40, variance 8.

(b) A geometric distribution with mean 2.

(c) A Poisson distribution with variance 4.

(d) An exponential distribution with mean 10.

(e) A gamma distribution with mean 6, variance 12.

(f) A normal distribution with mean 5 and variance 16.

(5) B06. Let X have an exponential distribution with mean $\alpha > 0$. Show that $\mathrm{P}(X > x + y | X > x) = \mathrm{P}(X > y)$ for any $x, y > 0$.

(5) B07. Let X have the p.d.f. $f(x) = \theta x^{\theta-1}$, $0 < x < 1$, $0 < \theta < \infty$, and let $Y = -2\theta ln X$.

(a) How is Y distributed?

(b) What is the moment-generating function of Y?

(5) B08. Let a r.v. X have the p.d.f. $f(x) = (x+1)/2, \ -1 < x < 1$. Find (a) $q_1 = \pi_{0.25}$, (b) $q_2 = \pi_{0.50}$, and (c) $q_3 = \pi_{0.75}$.

(5) B09. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from Poisson distribution with mean 2. Define $Y = \sum_{i=1}^{n} X_i$

(a) Find the moment-generating function of X_1.

(b) Find the moment-generating function of Y.

(c) How is Y distributed?

(10) B10. Let X, Y be a random sample of size 2 from $\sim N(3, 0.25)$. Define $Z = 2(X-3)$, $U = 2(Y-3)$, $W = Z^2$, $V = Z + U$.

(a) Write down the probability density function of X.

(b) Show that Z has the standard normal distribution.

(c) What is the moment-generating function of Z?

(d) Show that $W \sim \chi^2(1)$.

(e) What is the moment-generating function of W?

(f) What is the moment-generating function of V?

(g) How is V distributed?

(h) What is the probability density function of V?

(5) B11. Let X have an exponential distribution with a mean of $\theta = 20$. Compute

(a) $\mathrm{P}(10 < X < 30)$

(b) $\mathrm{P}(X > 30)$

(c) $\mathrm{P}(X > 40 | X > 10)$

(5) B12. Let X have a logistic distribution with p.d.f.

$$f(x) = e^{-x}/(1+e^{-x})^2, \; -\infty < x < \infty$$

Show that $Y = 1/(1+e^{-x}) \;\sim\; U(0,1)$

(Hint) Show that $P(Y \leq y) = y$ for $y \in (0,1)$.

B13. If the moment-generating function of X is $M(t) = (1-2t)^{-12}$, $t < 1/2$. Find

(a) $\mu = E(X)$

(b) $\sigma^2 = Var(X)$

(c) $\mathrm{P}(15.66 < X < 42.98)$

B14. If $X \sim N(\mu, \sigma^2)$, show that $Y = (aX + b) \sim N(a\mu + b, a^2\sigma^2), \;\; a \neq 0$.

B15. Let X have the p.d.f. $f(x) = \theta x^{\theta-1}$, $0 < x < 1$, $0 < \theta < \infty$, and let $Y = -2\theta lnX$. How is Y distributed? What is the moment-generating function of Y?

B16. Let $X_1 \sim b(n_1, p)$ and $X_2 \sim b(n_2, p)$ be independent r.v.'s. Define $Y = X_1 + X_2$.

(a) What is $M_Y(t)$?

(b) How is Y distributed?

Partial Solutions

B01(a) $\mathrm{P}(X = 4) = (e^{-4}4^4)/(4!) \approx 0.1954$

(b) $\mathrm{P}(2 \le X \le 6) \approx 0.889$ - $0.092 = 0.787$

B02(a) $M(t) = \frac{pe^t}{1-(1-p)e^t}$, $t < -ln(1-p)$

(b) $M(t) = (1 - p + pe^t)^n$

(c) $M(t) = e^{\lambda(e^t - 1)}$

B03(a) $\mu = \pi/2$, $\sigma^2 = (\pi^2 - 8)/4$

(b) Draw $(x,\, y = f(x))$, for $0 < x < \pi$.

(c) Draw $(x,\, y = F(x))$, where $F(x) = (1 - cos(x))/2$.

B04(a) $\pi_{0.25} = 0$

(b) $\pi_{0.50} = -1 + \sqrt{2} \approx 0.414$

(c) $\pi_{0.75} = -1 + \sqrt{3} \approx 0.732$

B05(a) $f(x) = C(50, 0.8)(0.8)^x(0.2)^{50-x}, \;\; 0 \le x \le 50.$

(b) $f(x) = (0.5)^x, \;\; x = 1, 2, \ldots.$

(c) $f(x) = e^{-4}4^x/(x!), \;\; x = 0, 1, \ldots.$

(d) $f(x) = \frac{1}{10}e^{-x/10}, \;\; 0 < x < \infty.$

(e) $f(x) = \frac{1}{\Gamma(3)2^3}x^2e^{-x/2}, \;\; 0 < x < \infty.$

(f) $f(x) = \frac{1}{4\sqrt{2\pi}}e^{-(x-5)^2/32}, \;\; -\infty < x < \infty.$

B07(a) Y has an exponential distribution with mean 2.

(b) $M_Y(t) = 1/(1-2t), \quad -1/2 < t < 1/2.$

B08(a) $\pi_{0.25} = 0$

(b) $\pi_{0.50} = -1 + \sqrt{2} \approx 0.414$

(c) $\pi_{0.75} = -1 + \sqrt{3} \approx 0.732$

B09(a) $M_{X_1}(t) = e^{2(e^t-1)}, \;\; t < 0(?).$

(b) $M_Y(t) = e^{2n(e^t-1)}, \;\; t < 0(?).$

(c) Y has a *Poisson* distribution with mean $2n$.

B10(a) $f(x) = \frac{2}{\sqrt{2\pi}} e^{-2(x-3)^2}, \quad -\infty < x < \infty.$

(c) $M_Z(t) = e^{t^2/2}.$

(e) $M_W(t) = \frac{1}{\sqrt{1-2t}}.$

(f) $M_V(t) = \exp[t^2].$

(g) $V \sim N(0, 2).$

(h) $f_V(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}, \quad -\infty < x < \infty.$

B11(a) $e^{-0.5} - e^{-1.5}$

(b) $e^{-1.5}$

(c) $e^{-1.5} = P(X > 30)$

B12. $P(Y \leq y) = P(1/(1+e^{-X}) \leq y) \Rightarrow P[X \leq ln(y/(1-y))]$

B13(a) $\mu = 24$ since $X \sim \chi^2(r = 24)$

(b) $\sigma^2 = 2r = 48$

(c) $\mathrm{P}(15.66 < X < 42.98) = 0.99 - 0.10 = 0.89$

B15. Y has an exponential distribution with mean $\theta = 2$, and $M_Y(t) = 1/(1-2t)$.

B16(a) $M_Y(t) = ((1-p) + pe^t)^{n-1+n_2}$.

(b) $Y \sim b(n_1 + n_2, p)$

Exam III for CS3332

January 4, 2002

C01. Show that the sum of n independent Poisson random variables with respective means $\lambda_1, \lambda_2, \ldots, \lambda_n$ is Poisson with mean $\lambda = \sum_{i=1}^{n} \lambda_i$.

C02. Let $Z_i \sim N(0,1)$, for $1 \leq i \leq 7$ and define $W = \sum_{i=1}^{7} Z_i^2$. Find P(1.69< W <14.07).

C03. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$ where $\sigma^2 > 0$ is known. Show that the maximum likelihood estimator for μ is $\widehat{\mu} = \overline{X}$.

C04. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$ where μ is known. Show that the maximum likelihood estimator for σ^2 is $\widehat{\sigma^2} = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2$.

C05. Find the maximum likelihood estimates for μ and σ^2 if a random sample of size n from $N(\mu, \sigma^2)$ yields the following values.

31.5	36.5	33.8	30.1	33.9
35.2	29.6	34.4	30.5	34.2
31.6	36.7	35.8	34.5	32.7

C06. For the following statements, mark a ◯ if it is *true*, and mark a $\times$ otherwise.

(a) Let $X_i \sim b(10, 0.8)$, $1 \le i \le 8$, be a random sample of size 8. Then $\sum_{i=1}^{8} X_i \sim b(80, 0.1)$.

(b) Let X_1, X_2, X_3, X_4 be mutually independent r.v.'s with Poisson distributions having variances 1, 2, 3, 4, respectively. Then $\sum_{i=1}^{4} X_i$ has a Poisson distribution with variance $1^2 + 2^2 + 3^2 + 4^2 = 30$.

(c) Let $X_i \sim N(1, 4)$, $1 \le i \le 4$, be a random sample of size 4. Then $\sum_{i=1}^{4} X_i/4 \sim N(1, 4)$.

(d) Let X have the standard normal distribution, α, $\beta > 0$ be constant. Then $(\alpha X + \beta) \sim N(\alpha, \beta^2)$.

(e) Let $\{Z_i\}$ be a random sample of size n from the standard normal distribution. Then $\overline{Z} = \frac{1}{n} \sum_{i=1}^{n} Z_i \sim N(0, 1)$.

(f) Let $\{Z_i\}$ be a random sample of size n from the standard normal distribution. Then $\sum_{i=1}^{n} Z_i^2 \sim \chi^2(n)$.

(g) Let Y have a Poisson distribution with mean $3n$. According to the central limit theorem, $(Y - 3n)/\sqrt{3n}$ has a limiting distribution $N(0, 1)$.

(h) Let $Y \sim \chi^2(n)$. According to the central limit theorem, $(Y - n)/\sqrt{2n}$ has a limiting distribution $N(0, 1)$.

(i) Let $\{X_i\}$, $1 \le i \le n$, be a random sample of size n, from an exponential distribution with mean 2. Let $Y = \sum_{i=1}^{n} X_i$. According to the central limit theorem, $(Y - 2n)/\sqrt{4n}$ has a limiting distribution $N(0, 1)$.

(j) Let $X \sim N(2, 2)$. Then $M_X(t) = e^{2t+t^2}$.

C07. Let $Z_1, Z_2, \ldots, Z_{10}$ be a random sample from $N(0,1)$ and let $W = \sum_{j=1}^{10} Z_j^2$. Find $\mathrm{P}(3.94 < W \leq 18.31)$.

C08. Let *random*() be a pseudo random number generator which can randomly generate a real in [0, 1). Let $X \sim N(0,1)$, and $Y \sim N(3,16)$. Give algorithms to sample (simulate) the distributions of X and Y, respectively.

C09. Let $X_1, X_2, \ldots, X_{30}$ be a random sample of size 30 from a Poisson distribution with a mean 2/3. Approximate

(a) $P(15 < \sum_{i=1}^{30} X_i \leq 22)$.

(b) $P(21 \leq \sum_{i=1}^{30} X_i < 27)$.

C10. Let R have a χ^2 distribution with 12 degrees of freedom. Find $P(6.304 < R \leq 18.55)$.

C11. A random sample of size 8 from $N(\mu, 72)$ yielded $\overline{x} = 85$. Find a 95% confidence interval for μ.

C12. Let $f(x;\theta) = (1/\theta)x^{(1-\theta)/\theta}$, $0 < x < 1,\ 0 < \theta < \infty$.

(a) Find the maximum likelihood estimator $\widehat{\theta}$ of θ.

(b) Show that $\widehat{\theta}$ is an unbiased estimator of θ.

Partial Solutions

C01. Let $X_i \sim Poisson(\lambda_i)$, then $M_{X_i}(t) = e^{\lambda_i(e^t-1)}$. Define $Y = \sum_{i=1}^n X_i$, then $M_Y(t) = e^{\lambda(e^t-1)}$, where $\lambda = \sum_{i=1}^n \lambda_i$, which completes the proof.

C02. Since $W \sim \chi^2(7)$, then $P(1.69 < W < 14.07) = 0.95$ - $0.025 = 0.925$.

C05. $\mu = (31.5 + 36.5 + 33.8 + \ldots + 34.5 + 32.7)/15 = 33.4$.

$\sigma^2 = \frac{1}{15}\left[(31.5 - 33.4)^2 + (36.5 - 33.4)^2 + \ldots + (32.7 - 33.4)^2\right]/15 = 4.9226$.

C06. For the following statements, mark a $\bigcirc$ if it is *true*, and mark a $\times$ otherwise. *(a)* $\times$, *(b)* $\times$, *(c)* $\times$, *(d)* $\times$, *(e)* $\times$,
(f) $\bigcirc$, *(g)* $\bigcirc$, *(h)* $\bigcirc$, *(i)* $\bigcirc$, *(j)* $\bigcirc$.

C07. P$(3.94 < W < 18.31) = 0.95$ - $0.05 = 0.90$

C08. Give algorithms to simulate $U(0, 1)$ and $N(0, 1)$.

C09(a) $P(15 < \sum_{i=1}^{30} X_i \leq 22) = 0.5548$

(b) $P(21 \leq \sum_{i=1}^{30} X_i < 27) = 0.3823$

C10. $R \sim \chi^2(12)$, $P(6.304 < R < 18.55) = 0.90$ - $0.10 = 0.80$

C11. A 95% confidence interval for μ is $[85 - 3 * 1.96,\ 85 + 3 * 1.96] = [79.12,\ 90.88]$

C12. #6.1.8 on *page 292*, 1997 ed.