# Order Statistics

□ If $X_1, X_2, \cdots, X_n$ are observations of a random sample of size $n$ from a continuous-type distribution whose p.d.f.is $f(x)$ and c.d.f. is $F(x)$, and let the random variables

$$Y_1 < Y_2 < \cdots < Y_n \quad or \quad X_{(1)} < X_{(2)} < \cdots < X_{(n)} \tag{1}$$

denote the order statistics of this sample, that is,

$$Y_1 \quad is \quad the\ smallest\ of\ X_1, X_2, \cdots, X_n$$

$$\vdots$$

$$Y_r \quad is \quad the\ r-th\ smallest\ of\ X_1, X_2, \cdots, X_n$$

$$\vdots$$

$$Y_n \quad is \quad the\ largest\ of\ X_1, X_2, \cdots, X_n$$

Let $f$ be defined in $(a, b)$ so that $F'(x) = f(x),\ for\ x \in (a, b),\ 0 < F(x) < 1,\ x \in (a, b)$ and $F(a) = 0,\ F(b) = 1$. Then we have

$$\begin{aligned} G_r(y) \quad = \quad P(Y_r \le y) \quad &= \textstyle\sum_{k=r}^{n} \begin{pmatrix} n \\ k \end{pmatrix} [F(y)]^k [1 - F(y)]^{n-k} \\ &= \textstyle\sum_{k=r}^{n-1} \begin{pmatrix} n \\ k \end{pmatrix} [F(y)]^k [1 - F(y)]^{n-k} + [F(y)]^n \end{aligned}$$

Thus the p.d.f. of $Y_r$ could be derived as

$$g_r(y) = {G_r}'(y) = \frac{n!}{(r-1)!(n-r)!} [F(y)]^{r-1} [1 - F(y)]^{n-r} f(y), \quad a < y < b \tag{2}$$

In particular,

$$\begin{aligned} g_1(y) &= n[1 - F(y)]^{n-1} f(y), \quad a < y < b \\ g_n(y) &= n[F(y)]^{n-1} f(y), \qquad\quad a < y < b \end{aligned}$$

**(1)** If $X_i$ has a U(0,1) distribution, $E(Y_r) = \frac{r}{n+1}$.

**(2)** If $X_j$ has an exponential distribution with mean 2, $g_1(y) = ne^{-ny},\ y > 0$ and $E(Y_1) = \frac{1}{n}$.