Chapter 5. Distributions of Functions of Random Variables

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Distributions of Functions of Random Variables

- We discuss the distributions of functions of one random variable $X$ and the distributions of functions of independently distributed random variables in this Chapter.

Example 1. Let $X$ have the p.d.f. $f(x) = xe^{-x^2/2}$, $0 < x < \infty$. Then $Y = X^2$ has an exponential distribution with mean 2.

Example 2. The p.d.f. of $X$ is $f(x) = \theta x^{\theta-1}$, $0 < x < 1$, $0 < \theta < \infty$. Then $Y = -2\theta \ln(X)$ has an exponential distribution with mean 2.

Example 3. Let $X$ have a logistic distribution with p.d.f.

$$f(x) = \frac{e^{-x}}{(1 + e^{-x})^2}, \quad -\infty < x < \infty$$

Then $Y = \frac{1}{1+e^{-X}}$ has a $U(0,1)$ distribution.

Example 4. Let $X_1 \sim b(m, p)$ and $X_2 \sim b(n, p)$ be independent r.v.’s, then $Y = X_1 + X_2 \sim b(m+n, p)$. 
Sampling Distribution Theory

♣ The collection of $n$ independent and identically distributed random variables $X_1, X_2, \ldots, X_n$, is called a random sample of size $n$ from the common distribution, e.g., $X_j \sim N(0, 1)$, $1 \leq j \leq n$.

♣ Some functions of a random sample, called statistics, are of interest, for examples, mean and variance. Sampling distribution theory refers to the derivation of distributions for functions of a random sample.

**Theorem 1:** Let $X_1, X_2, \ldots, X_n$ be $n$ independent r.v.'s with respective means $\{\mu_i\}$ and variances $\{\sigma_i^2\}$, then $Y = \sum_{i=1}^{n} a_i X_i$ has mean $\mu_Y = \sum_{i=1}^{n} a_i \mu_i$ and variance $\sigma_Y^2 = \sum_{i=1}^{n} a_i^2 \sigma_i^2$, respectively.

**Theorem 2:** Let $X_1, X_2, \ldots, X_n$ be $n$ independent r.v.'s with respective moment-generating functions $\{M_i(t)\}$, $1 \leq i \leq n$, then the moment-generating function of $Y = \sum_{i=1}^{n} a_i X_i$ is $M_Y(t) = \prod_{i=1}^{n} M_i(a_i t)$.

**Corollary:** If $X_1, X_2, \ldots, X_n$ are observations of a random sample from a distribution with moment-generating function $M(t)$, then

(a) $M_Y(t) = \prod_{i=1}^{n} M(t)$, where $Y = \sum_{i=1}^{n} X_i$.

(b) $M_{\bar{X}}(t) = \prod_{i=1}^{n} M(t/n)$, where $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$.

**Example 1:** Let $X_i \sim b(k, p)$ be a random sample of size $n$. Define $Y = \sum_{i=1}^{n} X_i$, then $M_Y(t) = \prod_{i=1}^{n} (q + pe^t)^k = (q + pe^t)^{kn}$.

**Example 2:** Let $X_i \sim Gamma(1, \theta)$ be a random sample of size $n$. Define $Y = \sum_{i=1}^{n} X_i$, then $M_Y(t) = \prod_{i=1}^{n} (1 - \theta t)^{-1} = 1/(1 - \theta t)^n$.

**Exercises:**
Random Functions Associated with Normal Distributions

In statistical applications, it is usually assumed that the population from which a sample is taken is $N(\mu, \sigma^2)$.

**Theorem:** Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from $N(\mu, \sigma^2)$. Define $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$, then $\bar{X} \sim N(\mu, \sigma^2/n)$.

**Theorem:** Let $X_j \sim \chi^2(r_j)$, $1 \leq j \leq n$. If $X_1, X_2, \ldots, X_n$ are independent, then $Y = \sum_{i=1}^{n} X_i \sim \chi^2(r_1 + r_2 + \ldots + r_n)$.

**Theorem:** Let $Z_1, Z_2, \ldots, Z_n$ be a random sample of size $n$ from $N(0, 1)$, then $W = Z_1^2 + Z_2^2 + \ldots + Z_n^2 \sim \chi^2(n)$.

**Corollary:** Let $\{X'_i\}$ be independent random variables from $N(\mu_i, \sigma_i^2)$, respectively, then $W = \sum_{i=1}^{n} (X_i - \mu_i)^2/\sigma_i^2$ is $\chi^2(n)$.

**Theorem:** Let $\{X'_i\}$ be observations of a random sample of size $n$ from $N(\mu, \sigma^2)$. Define $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$, then

(a) $\bar{X}$ and $S^2$ are independent.

(b) $\frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n} (X_i - \bar{X})^2/\sigma^2 \sim \chi^2(n-1)$.

**Example 1:** Let $X_1, X_2, X_3, X_4$ be a random sample of size 4 from the normal distribution $N(76.4, 383)$. Then

(a) $U = \sum_{i=1}^{n} (X_i - 76.4)^2/383 \sim \chi^2(4)$, $P(0.711 \leq U \leq 7.779) = 0.90 - 0.05 = 0.85$.

(b) $W = \sum_{i=1}^{n} (X_i - \bar{X})^2/383 \sim \chi^2(3)$, $P(0.352 \leq W \leq 6.251) = 0.90 - 0.05 = 0.85$.

**Theorem:** Let $X_i \sim N(\mu_i, \sigma_i^2)$, $1 \leq i \leq n$, be independent. Define $Y = \sum_{i=1}^{n} a_i X_i$, then $Y \sim N(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2)$.
The Central Limit Theorem

**Theorem:** Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from $N(\mu, \sigma^2)$. Define $\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$, then $\overline{X} \sim N(\mu, \sigma^2/n)$.

**Theorem:** Let $\overline{X}$ be the mean of a random sample $X_1, X_2, \ldots, X_n$ of size $n$ from a distribution with mean $\mu$ and variance $\sigma^2$. Define $W_n = (\overline{X} - \mu)/(\sigma/\sqrt{n})$. Then

(a) $W_n = (\sum_{i=1}^{n} X_i - n\mu)/(\sqrt{n}\sigma)$

(b) $P(W_n \leq w) \approx \int_{-\infty}^{w} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \Phi(w)$.

(c) $W_n \sim N(0, 1)$ as $n \to \infty$.

**Example 1:** Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from a $\chi^2(1)$. Define $Y = \sum_{i=1}^{n} X_i$. Then

(a) $Y \sim \chi^2(n)$.

(b) $(Y - n)/\sqrt{2n} \approx N(0, 1)$.

**Example 2:** Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from a $U(0, 1)$. Define $Y = \sum_{i=1}^{n} X_i$. Then

$$(Y - 0.5n)/\sqrt{n/12} \approx N(0, 1).$$

**Example 3:** Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from a $Bernoulli(p)$. Define $Y = \sum_{i=1}^{n} X_i$. Then

(a) $Y \sim b(n, p)$.

(b) $(Y - np)/\sqrt{np(1 - p)} \approx N(0, 1)$.

**Example 4:** Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from an exponential distribution with mean $\theta$. Define $Y = \sum_{i=1}^{n} X_i$. Then

(a) $Y \sim Gamma(n, \theta)$.

(b) $(Y - n\theta)/\sqrt{n\theta^2} \approx N(0, 1)$.
Approximations for Discrete Distributions

ymbol Use the normal distribution to approximate probabilities for certain discrete-type distributions.

Example 1: Let \( Y \sim b(10, 1/2) \). Then

\[
P(3 \leq Y < 6) = P(2.5 \leq Y \leq 5.5) = \Phi(0.316) - \Phi(-1.581) = 0.6240 - 0.0570 = 0.5670 \text{ (by Table II)}. \tag{1}
\]

Example 2: Let \( X_1, X_2, \ldots, X_n \) be a random sample of size \( n \) from a \( Poisson(\lambda) \). Define \( Y = \sum_{i=1}^{n} X_i \). Then

\[
(Y - n\lambda)/\sqrt{n\lambda} \approx N(0, 1).
\]

Example 3: Let \( Y \sim Poisson(\lambda = 20) \). Then

\[
P(16 < Y \leq 21) = P(16.5 \leq Y \leq 21.5) = P[(16.5 - 20)/\sqrt{20} \leq (Y - 20)/\sqrt{20} \leq (21.5 - 20)/\sqrt{20}]
= \Phi(0.335) - \Phi(-0.783) = 0.4142 \tag{2}
\]
Limiting Moment-Generating Functions

**Theorem:** If a sequence of moment-generating functions approaches a certain one, say, \( M(t) \), then the limit of the corresponding distribution must be the distribution corresponding to \( M(t) \).

**Example 1:** Let \( Y \sim b(50, 0.04) \) and let \( \lambda = np = 50 \times 0.04 = 2 \). Then

\[
P(Y \leq 1) = 0.400
\]

\[
P(Y \leq 1) \approx 0.406 \text{ by a Poisson approximation.}
\]
Simulating Continuous Distributions

**Theorem 5.1-2** Let $X$ have the cumulative distribution function (c.d.f.) $F(x)$ of the continuous type that is strictly increasing (i.e., $F(t) > F(s)$ if $t > s$) in on the support $a < x < b$. Then the r.v. $Y = F(X)$ has a uniform distribution $U(0, 1)$.

**Proof** Since $F(a) = 0$ and $F(b) = 1$. For $0 < y < 1$, we have

$$P(Y \leq y) = P(F(x) \leq y) = P(X \leq F^{-1}(y)) = F(F^{-1}(y)) = y$$

Thus, $Y$ has a uniform distribution $U(0, 1)$.

- Simulating an exponential distribution $f(x) = \frac{1}{2}e^{-x/2}$, $0 < x < \infty$.

1. $Y = F(X) = 1 - e^{-X/2} \sim U(0, 1)$,
2. Generate a $y$ from $U(0, 1)$ and let $y = 1 - e^{-x/2}$
3. Then $x = -2 \times \ln(1 - y + \epsilon)$,
4. Repeat steps (2) and (3) for the sample size you request.
Example 5.2-6: Box-Muller Transformation

**Box-Muller Transformation** Let \( \{X_1, X_2\} \) be a random sample from \( U(0,1) \), define

\[
Z_1 = \sqrt{-2 \ln X_1} \cos(2\pi X_2) \quad \text{and} \quad Z_2 = \sqrt{-2 \ln X_1} \sin(2\pi X_2).
\]

or, equivalently

\[
X_1 = \exp \left( -\frac{Z_1^2 + Z_2^2}{2} \right) = e^{-q/2} \quad \text{and} \quad X_2 = \frac{1}{2\pi} \arctan \left( \frac{Z_2}{Z_1} \right),
\]

which has the Jacobian

\[
J = \begin{vmatrix}
-z_1 e^{-q/2} & -z_2 e^{-q/2} \\
-\frac{z_2^2}{2\pi (z_1^2 + z_2^2)} & -\frac{z_1^2}{2\pi (z_1^2 + z_2^2)}
\end{vmatrix} = -\frac{1}{2\pi} e^{-q/2}.
\]

Since the joint p.d.f. of \( X_1 \) and \( X_2 \) is \( f(x_1, x_2) = 1, \ 0 < x_1, \ x_2 < 1 \), hence the joint p.d.f. of \( Z_1 \) and \( Z_2 \) is

\[
g(z_1, z_2) = |J_{x_1, x_2}| = \frac{1}{2\pi} \exp \left[ -(z_1^2 + z_2^2)/2 \right], \quad -\infty < z_1, \ z_2 < \infty.
\]
The Beta, Student’s t, and F Distributions

Random variables whose space are intervals or a union of intervals are said to be of the continuous types. The p.d.f. of a r.v. X of continuous type is an integrable function \( f(x) \) satisfying

(a) \( f(x) > 0, \ x \in R \)

(b) \( \int_R f(x)dx = 1 \)

(c) The probability of the event \( X \in A \) is \( P(A) = \int_A f(x)dx \)

**Beta** \( f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}, \ 0 < x < 1; \ \alpha, \beta > 0 \)

**Student’s t** Let \( Z \sim N(0,1) \) and \( V \sim \chi^2(r) \) be two independent random variables. Define \( T = Z/\sqrt{V/r} \). Then \( T \) has a t-distribution with p.d.f.

\[
f(t) = \frac{\Gamma((r+1)/2)}{\sqrt{\pi r} \Gamma(r/2)} \frac{1}{(1+t^2/r)^(r+1)/2}, \ \ -\infty < t < \infty
\]

**F-distribution** Let \( U \sim \chi^2(r_1) \) and \( V \sim \chi^2(r_2) \) be two independent random variables. Define \( W = (U/r_1)/(V/r_2) \). Then \( W \) has an F-distribution with p.d.f.

\[
f(w) = \frac{\Gamma((r_1+r_2)/2)(r_1/r_2)^{1/2}}{\Gamma(r_1/2)\Gamma(r_2/2)} \frac{w^{(r_1/2)-1}}{(1+w)^(r_1+r_2)/2}, \ \ 0 < w < \infty
\]
Proof of the Central Limit Theorem

**Theorem:** Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from $N(\mu, \sigma^2)$. Define $\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$, then $\overline{X} \sim N(\mu, \sigma^2/n)$.

**Theorem:** Let $\overline{X}$ be the mean of a random sample $X_1, X_2, \ldots, X_n$ of size $n$ from a distribution with mean $\mu$ and variance $\sigma^2$. Define $W_n = (\overline{X} - \mu) / (\sigma / \sqrt{n})$. Then

(a) $W_n = (\sum_{i=1}^{n} X_i - n\mu) / (\sqrt{n}\sigma)$

(b) $P(W_n \leq w) \approx \int_{-\infty}^{w} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \Phi(w)$.

(c) $W_n \sim N(0, 1)$ as $n \to \infty$.

**(Proof)**

$$E[\exp(tW_n)] = E\{\exp\left[\left(\frac{t}{\sqrt{n}\sigma}\right)(\sum_{i=1}^{n} X_i - n\mu)\right]\}$$

$$= E\{\exp\left[\left(\frac{t}{\sqrt{n}}\right)\left(\frac{X_1 - \mu}{\sigma}\right) + \cdots + \left(\frac{t}{\sqrt{n}}\right)\left(\frac{X_n - \mu}{\sigma}\right)\right]\}$$

$$= E\{\exp\left[\left(\frac{t}{\sqrt{n}}\right)\left(\frac{X_1 - \mu}{\sigma}\right)\right]\} \cdots E\{\exp\left[\left(\frac{t}{\sqrt{n}}\right)\left(\frac{X_n - \mu}{\sigma}\right)\right]\},$$

which follows from the independence of $X_1, X_2, \cdots, X_n$. Then

$$E[\exp(tW_n)] = \left[M\left(\frac{t}{\sqrt{n}}\right)\right]^{n}, \quad -h < \frac{t}{\sqrt{n}} < h,$$

where

$$M(t) = E\{\exp\left[t\left(\frac{X_i - \mu}{\sigma}\right)\right]\}, \quad -h < t < h$$

is the common moment-generating function of each

$$Y_i = \frac{X_i - \mu}{\sigma}, \quad i = 1, 2, \ldots, n.$$ 

since $E(Y_i) = 0$ and $E(Y_i^2) = 1$, it must be that

$$M(0) = 1, \quad M'(0) = E\left(\frac{X_i - \mu}{\sigma}\right) = 0, \quad M''(0) = E\left[\left(\frac{X_i - \mu}{\sigma}\right)^2\right] = 1$$

Hence, using Taylor’s formula with a remainder, we know that there exists a number $t_1$ between 0 and $t$ such that

$$M(t) = M(0) + M'(0)t + \frac{M''(t_1)t^2}{2} = 1 + \frac{M''(t_1)t^2}{2}. $$
Adding and subtracting $t^2/2$, we have

$$M(t) = 1 + \frac{t^2}{2} + \frac{1}{2}[M''(t_1) - 1]t^2.$$  

Using this expression of $M(t)$ in $E[\exp(tW_n)]$, we can represent the moment-generating function of $W_n$ by

$$E[\exp(tW_n)] = \left\{ 1 + \frac{1}{2} \left( \frac{t}{\sqrt{n}} \right)^2 + \frac{1}{2}[M''(t_1) - 1] \left( \frac{t}{\sqrt{n}} \right)^2 \right\}^n$$

$$= \left\{ 1 + \frac{t^2}{2n} + \frac{[M''(t_1) - 1]^2}{2n} \right\}^n, \quad -\sqrt{n}h < t < \sqrt{n}h,$$

where now $t_1$ is between 0 and $t/\sqrt{n}$. Since $M''(t)$ is continuous at $t = 0$ and $t_1 \to 0$ as $n \to \infty$, we have

$$\lim_{n \to \infty} [M''(t_1) - 1] = 1 - 1 = 0$$

Thus,

$$\lim_{n \to \infty} E[\exp(tW_n)] = \lim_{n \to \infty} \left\{ 1 + \frac{t^2}{2n} + \frac{[M''(t_1) - 1]^2}{2n} \right\}^n$$

$$= \lim_{n \to \infty} \left\{ 1 + \frac{t^2}{2n} \right\}^n = e^{t^2/2}$$

for all real $t$. We know that $e^{t^2/2}$ is the moment-generating function of the standard normal distribution, $N(0, 1)$. Therefore, the limiting distribution of

$$W_n = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma^2} \rightarrow N(0, 1) \quad \text{as} \quad n \to \infty.$$
Order Statistics

If \(X_1, X_2, \cdots, X_n\) are observations of a random sample of size \(n\) from a continuous-type distribution whose p.d.f. is \(f(x)\) and c.d.f. is \(F(x)\), and let the random variables

\[ Y_1 < Y_2 < \cdots < Y_n \quad \text{or} \quad X(1) < X(2) < \cdots < X(n) \]  

denote the order statistics of this sample, that is,

\[ Y_1 \text{ is the smallest of } X_1, X_2, \cdots, X_n \]

\[ : \]

\[ Y_r \text{ is the } r-th \text{ smallest of } X_1, X_2, \cdots, X_n \]

\[ : \]

\[ Y_n \text{ is the largest of } X_1, X_2, \cdots, X_n \]

Let \(f\) be defined in \((a, b)\) so that \(F'(x) = f(x)\), for \(x \in (a, b)\), \(0 < F(x) < 1\), \(x \in (a, b)\) and \(F(a) = 0\), \(F(b) = 1\). Then we have

\[ G_r(y) = P(Y_r \leq y) = \sum_{k=r}^{n} \binom{n}{k} [F(y)]^k [1 - F(y)]^{n-k} \]

\[ = \sum_{k=r}^{n-1} \binom{n}{k} [F(y)]^k [1 - F(y)]^{n-k} + [F(y)]^n \]

Thus the p.d.f. of \(Y_r\) could be derived as

\[ g_r(y) = G_r'(y) = \frac{n!}{(r-1)!(n-r)!} [F(y)]^{r-1} [1 - F(y)]^{n-r} f(y), \quad a < y < b \]  

In particular,

\[ g_1(y) = n[1 - F(y)]^{n-1} f(y), \quad a < y < b \]

\[ g_n(y) = n[F(y)]^{n-1} f(y), \quad a < y < b \]

(1) If \(X_i\) has a U(0,1) distribution, \(E(Y_r) = \frac{r}{n+1}\).

(2) If \(X_j\) has an exponential distribution with mean 2, \(g_1(y) = ne^{-ny}, \quad y > 0\) and \(E(Y_1) = \frac{1}{n}\).