Chapter 2. Discrete Distributions

♣ Random Variables of the Discrete Type

♣ Mathematical Expectation

♣ The Mean, Variance, and Standard Deviation

♣ Bernoulli Trials and the Binomial Distributions

♣ The Moment Generating Functions

♣ The Poisson Distribution

□ *Random Variables of the Discrete Type*

Definition: Given a random experiment with an outcome space S, a function X that assigns to each element s in S one and only one real number $x = X(s)$ is called a *random variable* (r.v.). The space of X is referred to as the set of real numbers $\Omega = \{x : \ X(s) = x, \ s \in S\}$.

Definition: The *probability mass function* (p.m.f) f of a discrete r.v. X is a function that satisfies the following properties:

(a) $f(x) > 0$, $x \in \Omega$;

(b) $\sum_{x\in\Omega} f(x) = 1$;

(c) $P(Y \subset \Omega) = \sum_{x\in Y} f(x)$;

Example 1. Roll a four-sided die twice, and let X equal the larger of the two outcomes are different and the common value if they are the same. The outcome space for the experiment is $S = \{(d_1, d_2)|1 \le d_1, d_2 \le 4\}$, where we assume that each of these 16 points has probability $\frac{1}{16}$. Then $P(X = 1) = P[(1, 1)] = 1/16$, $P(X = 2) = P[\{(1, 2), (2, 1), (2, 2)\}] = 3/16$, and similarly $P(X = 3) = 5/16$ and $P(X = 4) = 7/16$. The p.m.f. of X can be written simply as

$$f(x) = P(X = x) = \frac{2x - 1}{16}, \quad x = 1, 2, 3, 4.$$

Mathematical Expectation

Definition: If f is the p.m.f. of the r.v. X of the discrete type with space Ω and if the summation $\sum_{x\in\Omega} u(x)f(x)$ exists, then the sum is called the *mathematical expectation*, or the expected value of the function $u(X)$, which is denoted by $E[u(X)]$, that is,

$E[u(X)] = \sum_{x\in\Omega} u(x)f(x)$.

Theorem: The mathematical expectation E satisfies

(a) If c is a constant, $E[c] = c$.

(b) If c is a constant, and u is a function, then $E[cu(X)] = cE[u(X)]$.

(c) If c_1, c_2 are constants, and u_1, u_2 are functions, then $E[c_1u_1(X) + c_2u_2(X)] = c_1E[u_1(X)] + c_2E[u_2(X)]$.

Definition: The kth moment $m_k,\ k = 1, 2, \cdots$ of a random variable X is defined by the equation

$$m_k = E(X^k), \quad where\ k = 1, 2, \cdots$$

Then $E(X) = m_1$, and $Var(X) = E[(X-E(X))^2] = E(X^2)-[E(X)]^2 = m_2-(m_1)^2$.

Example 1. The number of defects on a printed board is a r.v. X with p.m.f. given by

$$P(X = i) = \frac{\gamma}{i+1}, \quad for\ i = 0, 1, 2, 3, 4$$

(a) Show that the constant $\gamma = \frac{60}{137}$.

(b) Show that $E(X) = \frac{163}{137}$ and $Var(X) = \frac{33300}{18769}$.

Example 2. The number of cells (out of 100) that exhibit chromosome aberrations is a random variable X with p.m.f. given by

$$P(X = i) = \frac{\beta(i+1)^2}{2^{i+1}}, \quad for\ i = 0, 1, 2, 3, 4, 5$$

(a) Show that the constant $\beta = \frac{32}{159}$.

(b) Show that $E(X) = \frac{390}{159}$ and $Var(X) = \frac{57462}{25281}$.

Uniform, Bernoulli, Binomial, Geometric, and Poisson Distributions

Uniform Distribution When a probability mass function (p.m.f.) is constant on the space or support, we say that the distribution is *uniform* over the space. For example, let X is the face value of rolling a fair die, then X has a discrete uniform distribution on $S = \{1, 2, 3, 4, 5, 6\}$ and its p.m.f. is

$$f(x) = \frac{1}{6}, \quad x = 1, 2, 3, 4, 5, 6.$$

Bernoulli Trials A r.v. X assuming only two values 0 and 1 with the probability P(X=1)=p and P(X=0)=q=$1-p$ is called a Bernoulli r.v. Each action is called a Bernoulli trial.

Binomial Distribution Consider a sequence of n independent Bernoulli trials. A r.v. X with the probability of exactly x successes is

$$\mathrm{f}(x) = \mathrm{P}(\mathrm{X}{=}x) = \mathrm{C(n,x)}p^x q^{n-x},\ x = 0, 1, \ldots, n.$$

Geometric Distribution Consider a sequence of independent Bernoulli trials. A r.v. X with the probability of the first success (X=1) at the $x - th$ trial equals

$$\mathrm{f}(x) = \mathrm{P}(\mathrm{X}{=}x) = q^{x-1}p,\ x = 1, 2, \ldots$$

Poisson Distribution A r.v. X has a Poisson distribution with parameter $\lambda > 0$ if

$$\mathrm{f}(x) = \mathrm{P}(\mathrm{X}{=}x) = (e^{-\lambda}\lambda^x)/(x!),\ x = 0, 1, \ldots$$

Approximate Poisson Process

For the number of changes that occurs in a given continuous interval, we have an *approximate Poisson process* with parameter $\lambda > 0$ if

(1) The number of changes occurring in nonoverlapping intervals are independent.

(2) The probability of *exactly one change* in a sufficient short interval of length Δ is approximated by $\lambda\Delta$.

(3) The probability of *two* or *more* changes in a sufficient short interval is *essentially* zero.

Let λ be fixed, and $\Delta = \frac{1}{n}$ with a large n.

$$\begin{aligned}
P(X = x) &= \binom{n}{x} (\tfrac{\lambda}{n})^x (1 - \tfrac{\lambda}{n})^{n-x} \\
&= \tfrac{n!}{(n-x)!x!} \tfrac{\lambda^x}{n^x} (1 - \tfrac{\lambda}{n})^n (1 - \tfrac{\lambda}{n})^{-x} \\
&= \tfrac{n}{n} \cdot \tfrac{n-1}{n} \cdots \tfrac{n-x+1}{n} \tfrac{\lambda^x}{x!} (1 - \tfrac{\lambda}{n})^n (1 - \tfrac{\lambda}{n})^{-x} \\
&= \tfrac{\lambda^x e^{-\lambda}}{x!} \quad as\ n \ \rightarrow \ \infty
\end{aligned}$$

Moment-Generating Functions

Definition: Let X be a r.v. of the discrete type with p.m.f. f and the sample space S. if there is an $h > 0$ such that

$$M(t) \equiv E(e^{tX}) = \sum_{x \in S} e^{tx} f(x)$$

exists and is finite for $t \in (-h,\ h)$, then the function *M(t)* is called the *moment-generating function (m.g.f.)* of X.

Remark: If the *m.g.f.* exists, there is one and only one distribution of probability associated with that *m.g.f.*

Binomial Distribution: For $X \sim b(n,p),\ \textit{and}\ p+q=1$,

$$\begin{aligned} M(t) &= E(e^{tX}) = \sum_{x=0}^{n} e^{tx} \begin{pmatrix} n \\ x \end{pmatrix} p^x q^{n-x} \\ &= \sum_{x=0}^{n} \begin{pmatrix} n \\ x \end{pmatrix} (pe^t)^x q^{n-x} = (q+pe^t)^n \end{aligned}$$

$E(X) = M'(0) = np$ and $Var(X) = M''(0) - [M'(0)]^2 = np(1-p)$.

Poisson Distribution: Let X have a Poisson distribution with mean λ, then

$$\begin{aligned} M(t) &= E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} \\ &= \sum_{x=0}^{\infty} e^{-\lambda} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} \cdot e^{\lambda e^t} = e^{\lambda(e^t-1)} \end{aligned}$$

$E(X) = M'(0) = \lambda$ and $Var(X) = M''(0) - [M'(0)]^2 = \lambda$.

Mean, Variance, and Moment Function of Discrete Distributions

Bernoulli $f(x) = p^x(1-p)^{1-x}, \quad x = 0, 1$

$$M(t) = 1 - p + pe^t; \qquad \mu = p, \ \sigma^2 = p(1-p)$$

Binomial $f(x) = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n$

$b(n,p) \qquad M(t) = (1 - p + pe^t)^n; \qquad \mu = np, \ \sigma^2 = np(1-p)$

Geometric $f(x) = (1-p)^{x-1}p, \quad x = 1, 2, \ldots$

$$M(t) = \frac{pe^t}{1-(1-p)e^t}, \quad t < -ln(1-p)$$

$$\mu = \frac{1}{p}, \ \sigma^2 = \frac{1-p}{p^2}$$

Hypergeometric $f(x) = \frac{C(N_1,x)C(N_2,n-x)}{C(N,n)}, \quad x \le n, x \le N_1, \, n-x \le N_2$

$$M(t) = \times$$

$$\mu = n\left(\frac{N_1}{N}\right), \ \sigma^2 = n\left(\frac{N_1}{N}\right)\left(\frac{N_2}{N}\right)\left(\frac{N-n}{N-1}\right)$$

Negative Binomial $f(x) = C(x-1, r-1)p^r(1-p)^{x-r}, \quad x = r, r+1, r+2, \ldots$

$$M(t) = \frac{(pe^t)^r}{[1-(1-p)e^t]^r} \quad t < -ln(1-p)$$

$$\mu = \frac{r}{p}, \ \sigma^2 = \frac{r(1-p)}{p^2}$$

Poisson $f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$

$$M(t) = e^{\lambda(e^t-1)}; \qquad \mu = \lambda, \ \sigma^2 = \lambda$$

Uniform $f(x) = \frac{1}{m}, \quad x = 1, 2, \ldots$

$$M(t) = \frac{1}{m} \cdot \frac{e^t(1-e^{mt})}{1-e^t}; \qquad \mu = \frac{m+1}{2}, \ \sigma^2 = \frac{m^2-1}{12}$$

Some Examples

1. Compute the probability function of the r.v. X that records the sum of the faces of two dice.

Solution: The sample space $\Omega = \{(i,j)|\ 1 \le i,j \le 6\}$. The random variable X is the function $X(i,j) = i+j$ which takes the range $R = \{2, 3, \cdots, 12\}$ with the probability function listed as

The probabilities of sum of two faces in casting two dice

$X(i,j) = s$	2	3	4	5	6	7
$P(X = s)$	1/36	2/36	3/36	4/36	5/36	6/36
$X(i,j) = s$	8	9	10	11	12	
$P(X = s)$	5/36	4/36	3/36	2/36	1/36	

2. It is claimed that 15% of the chickens in a particular region have patent H5N1 infection. Suppose seven chickens are selected at random. Let X equal the number of chickens that are infected.

(a) Assuming independence, how is X distributed? $[X \sim b(7, 0.15)]$.

(b) $P(X = 1) = \begin{pmatrix} 7 \\ 1 \end{pmatrix} (0.15)^1(0.85)^6$.

(c) $P(X \ge 2) = 1 - P(0) - P(1) = 1 - (0.85)^7 - \begin{pmatrix} 7 \\ 1 \end{pmatrix} (0.15)^1(0.85)^6$.

3. Let a r.v. X have a binomial distribution with mean 6 and variance 3.6. Find $P(X = 4)$.

Solution: Since $X \sim b(n,p)$ with $np = 6$ and $npq = 3.6$, then $q = 0.6$, $p = 0.4$, and $n = 15$. Thus, $P(X = 4) = \begin{pmatrix} 15 \\ 4 \end{pmatrix} (0.4)^4(0.6)^{11} \approx 0.1992$.

4. Let a r.v. X have a geometric distribution. Show that

$$P(X > k + j | X > k) = P(X > j), \quad where \ \ k, j \geq 0$$

We sometimes say that in this situation there has been loss of momory.

Solution: Let p be *the rate of success* in a geometric distribution. Then $P(X > j) = \sum_{r=j+1}^{\infty}(1-p)^{r-1}p = (1-p)^j$, thus

$$P(X > k + j | X > k) = \frac{P(X > k + j)}{P(X > k)} = \frac{(1-p)^{k+j}}{(1-p)^k} = (1-p)^j = P(X > j).$$

5. Let X have a Poisson distribution with a variance of 3, then $P(X = 2) = \frac{e^{-3}3^2}{2!} \approx 0.224$ and $P(X = 3) = \frac{e^{-3}3^3}{3!} \approx 0.224$.

6. Flaws in a certain type of drapery material appear on the average of one in 150 square feet. If we assume the Poisson distribution, find the probability of at most one flaw in 225 square feet.

Solution: Since $\lambda = 225/150 = 1.5$, then $P(X \leq 1) = \frac{e^{-1.5}(1.5)^0}{0!} + \frac{e^{-1.5}(1.5)^1}{1!} \approx 0.5578$

Negative Binomial Distribution

□ *Negative Binomial Distribution*

Suppose that we observe that a sequence of Bernoulli trials until *exactly* r successes occur. Let the random variable X denote the number of trials needed to observe the rth success. Let the probability of success is p in a Bernoulli trial. Then X has a negative binomial distribution with the p.m.f.

$$f(x) = \begin{pmatrix} x-1 \\ r-1 \end{pmatrix} p^r q^{x-r}, \quad x = r, r+1, r+2, \cdots$$

and

$$M(t) = \frac{(pe^t)^r}{[1-(1-p)e^t]^r}, \quad t < -ln(1-p)$$

$$E(X) = \frac{r}{p}, \quad and \quad Var(X) = \frac{r(1-p)}{p^2}$$

Note : $\quad (1-w)^{-r} = \sum_{k=0}^{\infty} \begin{pmatrix} k+r-1 \\ r-1 \end{pmatrix} w^k$

[**Proof**:]

$$\begin{aligned} M(t) &= \textstyle\sum_{x=r}^{\infty} e^{tx} \begin{pmatrix} x-1 \\ r-1 \end{pmatrix} p^r q^{x-r} &&= \textstyle\sum_{k=0}^{\infty} \begin{pmatrix} k+r-1 \\ r-1 \end{pmatrix} (p^r e^{tr})(q^k e^{tk}) \\ &= (pe^t)^r \textstyle\sum_{k=0}^{\infty} \begin{pmatrix} k+r-1 \\ r-1 \end{pmatrix} (qe^t)^k &&= (pe^t)^r (1-qe^t)^{-r} \\ &= \tfrac{(pe^t)^r}{(1-qe^t)^r}, \quad t < -ln(1-p) \end{aligned}$$

$$\begin{aligned} E(X) &= \textstyle\sum_{x=r}^{\infty} x \begin{pmatrix} x-1 \\ r-1 \end{pmatrix} p^r q^{x-r} &&= \textstyle\sum_{y=r+1}^{\infty} r \cdot \begin{pmatrix} y-1 \\ r \end{pmatrix} p^r q^{y-(r+1)} \\ &= rp^r(1-q)^{-r-1} &&= \tfrac{r}{p} \end{aligned}$$

$$Var(X) = E(X^2) - [E(X)]^2 = E(X(X-1)) + E(X) - [E(X)]^2$$

Hypergeometric Distribution

□ *Hypergeometric Distribution*
Consider a collection of N_1 green marbles and N_2 blue marbles of the same size, where $N_1 + N_2 = N$. A collection of n marbles is selected from N marbles at random *without replacement*. Let a r.v. X be the number of green marbles selected among n with $0 \le x \le n$. Then X has a hypergeometric distribution with the p.m.f.

$$f(x) = \frac{\begin{pmatrix} N_1 \\ x \end{pmatrix}\begin{pmatrix} N_2 \\ n-x \end{pmatrix}}{\begin{pmatrix} N \\ n \end{pmatrix}} \quad 0 \le x \le n \le N_1, \quad n-x \le N_2, \quad N_1 + N_2 = N.$$

and

$$E(X) = n\left(\frac{N_1}{N}\right), \quad \textit{and} \quad Var(X) = n\left(\frac{N_1}{N}\right)\left(\frac{N_2}{N}\right)\left(\frac{N-n}{N-1}\right)$$

What is $M(t)$? (*not available*)

Note : $(1+y)^N = (1+y)^{N_1}(1+y)^{N_2}$

[**Proof**:]

$$\begin{aligned} E(X) &= \textstyle\sum_{x=0}^{n} x \cdot \frac{\begin{pmatrix} N_1 \\ x \end{pmatrix}\begin{pmatrix} N_2 \\ n-x \end{pmatrix}}{\begin{pmatrix} N \\ n \end{pmatrix}} \\ &= \cdots \\ &= \textstyle\sum_{y=0}^{n-1} \left(\frac{n \cdot N_1}{N}\right) \frac{\begin{pmatrix} N_1 - 1 \\ y \end{pmatrix}\begin{pmatrix} N_2 \\ (n-1)-y \end{pmatrix}}{\begin{pmatrix} (N_1 - 1 + N2) = N-1 \\ n-1 \end{pmatrix}} \quad = \quad n\left(\frac{N_1}{N}\right) \end{aligned}$$

$$Var(X) = E(X^2) - [E(X)]^2 = E(X(X-1)) + E(X) - [E(X)]^2$$

Matlab Code and Results

```
% Script file: ch2.m - Discrete Distributions
  subplot(2,2,1)
  X=1:10; Y=geopdf(X,0.5); bar(X,Y,0.8);
  legend('Geometric Distribution: p=0.5',1)
  subplot(2,2,2)
  X=0:10; Y=poisspdf(X,3); bar(X,Y,0.8)
  legend('Poisson Distribution: \lambda =3',1)
  subplot(2,2,3)
  X=0:10; Y=binopdf(X,10,0.7); bar(X,Y,0.8)
  legend('X \sim b(10,0.7), mode=7',1)
  subplot(2,2,4)
  X=0:11; Y=binopdf(X,11,0.5); bar(X,Y,0.8)
  legend('X \sim b(11,0.5), mode=5,6',2)
```