Chapter 1. Probability

♣ Basic Concepts

♣ Properties of Probability

♣ Methods of Enumeration

♣ Conditional Probability

♣ Independent Events

♣ Bayes's Theorem

□ *Basic Concepts* The discipline of Statistics deals with the *collection* and *analysis* of data which is based on Probability Theory.

- Consider *Experiemnts* for which the outcome cannot be predicted with certainty, two definitions are given.
- S (Ω): Sample Space (outcome Space)
- E: An Event (a subset of outcome space)

♢ Example 1: Flipping a fair coin, $S = \{h, t\}$, $E = \{h\}$.

♢ Example 2: Sum of rolling a pair of two dice, $S = \{2, 3, 4, \cdots, 11, 12\}$, $E = \{2, 3, 4, 5\}$.

♢ Example 3: Scores (0∼100) of *96* students who take CS3332, $S = \{(i_1, x_1), (i_2, x_2), \cdots, (i_{96}, x_{96})\}$, $E = \{(i_j, x_j)|x_j < 60,\ 1 \le j \le 96\}$.

Properties of Probability

Notations $\emptyset$, $A \subset B$, $A \cup B$, $A \cap B$, A'

Definition: $A_1,\ A_2,\ \ldots, A_k$ are *mutually exclusive events* if $A_i \cap A_j = \emptyset$, $i \neq j$.

Definition: $A_1,\ A_2,\ \ldots, A_k$ are *exhaustive events* if $\bigcup_{i=1}^{k} A_i = S$.

Definition: *Probability* is a set function P that assigns to each event $A \subset S$ a number $P(A)$, called the probability of the event A, such that the following properties are satisfied:

(a) $P(A) \geq 0$,

(b) $P(S) = 1$,

(c) $P(\bigcup_{i=1}^{k} A_i) = \sum_{i=1}^{k} P(A_i)$ if $A_1,\ A_2,\ \ldots, A_k$ are *mutually exclusive events.*

(d) $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ for an infinite, but countable, number of events.

Property 1: For each event, $P(A') = 1 - P(A)$.

Property 2: $P(\emptyset) = 0$.

Property 3: $P(A) \leq P(B)$ if $A \subset B$, and $P(A) \leq 1$ for each event A.

Property 4: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Property 5: $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$.

Example 1: Suppose that $P(A) = 0.7,\ P(B) = 0.5$, and $P([A \cup B]^c) = 0.1$. Show that

(a) $P(A \cap B) = 0.3$, (b) $P(A|B) = \frac{3}{5}$, (c) $P(B|A) = \frac{3}{7}$.

Example 2: Let A and B be events such that $P(A) = \alpha$, $P(B) = \beta$, and $P(A \cup B) = \gamma$ are known. Express the following in terms of $\alpha,\ \beta,\ \gamma$.

(a) $P(A \cap B)$, (b) $P(A \cap B^c)$, (c) $P(B \cup [A \cap B^c])$, (d) $P(A^c \cap B^c)$.

Methods of Enumeration

Definition: Each of the $n!$ arrangements (in a row) of n different objects is called a *permutation* of the n objects.

Definition: Each of the ${}_nP_r = P(n,r)$ arrangements is called a *permutation* of the n objects taken r at a time.

Definition: Each of the ${}_nC_r = C(n,r)$ unordered subsets is called a *combination* of the n objects taken r at a time where ${}_nC_r = \begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r!(n-r)!}$

Definition: Since a binomial expansion can be written as $(a+b)^n = \sum_{r=0}^{n} C(n,r)a^r b^{n-r}$, so ${}_nC_r = \begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r!(n-r)!}$ is referred to as a binomial coefficient.

Definition: Multinomial coefficeints.

Definition: *Sampling with replacements* occurs when an object is selected and then replaced before the next object is selected.

Definition: *Sampling without replacements* occurs when an object is not replaced after it has been selected.

Example 1: $(a+b)^n = \displaystyle\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a^k b^{n-k}$

Example 2: (a) $\displaystyle\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} = 2^n$ and (b) $\displaystyle\sum_{k=0}^{n} \left\{ (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} \right\} = 0.$

Example 3: Show that $\begin{pmatrix} n \\ k \end{pmatrix} = \begin{pmatrix} n-1 \\ k \end{pmatrix} + \begin{pmatrix} n-1 \\ k-1 \end{pmatrix}.$

Probabilities of A Poker Hand

♣ A poker hand is defined as drawing 5 cards at random without replacement from a deck of 52 cards. The probability of each of the following poker hands can be computed.

(a) Four of a kind (four cards of equal face value and one card of a different value)

$$C(13,1) \times C(48,1)/C(52,5) \;=\; 624/2598960 \;=\; 0.00024.$$

(b) Full house (one pair and one triplet of cards with equal face value)

$$C(13,1) \times C(4,2) \times C(12,1) \times C(4,3)/C(52,5) \;=\; 3744/2598960 \;=\; 0.00144.$$

(c) Three of a kind

$$C(13,1) \times C(4,3) \times C(48,2)/C(52,5) \;=\; 54912/2598960 \;=\; 0.02113.$$

(d) Two pairs

$$C(13,2) \times C(4,2) \times C(4,2) \times C(44,1)/C(52,5) \;=\; 123552/2598960 \;=\; 0.047539.$$

(e) One pair

$$C(13,1) \times C(4,2) \times C(48,1) \times C(44,1) \times C(40,1)/C(52,5) \;=\; 1098240/2598960 \;=\; 0.42257.$$

Conditional Probability and Indepent Events

□ *Conditional Probability*

Definition: The *conditional probability* of an event A given that event B has occurred, is defined by

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ provided that $P(B) > 0$.

Property: $P(A \cap B) = P(B|A)P(A) = P(A|B)P(B)$.

Example 1: If $P(B)$ and $P(B \cap C)$ are both positive, derive the chain rule of conditional probability

$$P([A \cap B]|C) = P(A|[B \cap C])P(B|C).$$

Also show that

$$P(A \cap B \cap C) = P(A|[B \cap C])P(B|C)P(C).$$

□ *Independent Events*

Definition: Events A *and* B are *indepentent* iff $P(A \cap B) = P(A)P(B)$. Otherwise, A *and* B are dependent events.

Definition: Events A, B, *and* C are *mutually indepentent* iff

(a) $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, and $P(B \cap C) = P(B)P(C)$.

(b) $P(A \cap B \cap C) = P(A)P(B)P(C)$.

Example 1: If $P(A) = 0.8$, $P(B) = 0.5$, and $P(A \cup B) = 0.9$, are A and B independent events? Why?

Example 2: Suppose that A,B, *and* C are mutually indepentent events and that $P(A) = 0.5$, $P(B) = 0.8$, and $P(C) = 0.9$. Find the probabilities that

(a) All three events occur.

(b) Exactly two of the three events occur.

(c) None of the events occurs.

More Examples about Conditional Probabilities

(1) A box contains seven blue balls and three red balls. Two balls are to be drawn successively at random and without replacement. We want to compute the probability that the first draw results in a red ball (Event A) and the second draw iresults in a blue ball (Event B).

$$P(A) = \frac{3}{10}, \quad P(B|A) = \frac{7}{9}, \quad P(A \cap B) = \frac{3}{10} \cdot \frac{7}{9} = \frac{7}{30}$$

(2) A box contains four marbles numbered 1 through 4. The marbles are selected one at a time without replacement. A match occurs if marble numbered k is the kth marble selected. Let the event A_i, denote a match on the ith draw, i=1,2,3,4.

(a) Find P(A_i), for i=1,2,3,4.

(b) Find P($A_i \cap A_j$), where $1 \leq i < j \leq 4$.

(c) Find P($A_i \cap A_j \cap A_k$), where $1 \leq i < j < k \leq 4$.

(d) Find P($A_1 \cup A_2 \cup A_3 \cup A_4$).

Solutions

(a) P(A_i) = $3!/4!$, for i=1,2,3,4.

(b) P($A_i \cap A_j$) = $2!/4!$, where $1 \leq i < j \leq 4$.

(c) P($A_i \cap A_j \cap A_k$) = $1!/4!$, where $1 \leq i < j < k \leq 4$.

(d) P($A_1 \cup A_2 \cup A_3 \cup A_4$) = $1 - 1/2! + 1/3! - 1/4! = \frac{5}{8}$.

Bayes Rule and its Applications

Bayes Rule: $P(B_k|A) = P(A|B_k)P(B_k)/\sum_{i=1}^{n} P(A|B_i)P(B_i)$

Example 1: In a certain factory, machines A, B, *and* C are all producing springs of the same length. Of their production, machines A, B, *and* C produce 2%, 1%, and 3% defective springs, respectively. Of the total production of springs in the factory, machine A produces 35%, machine B produces 25%, and machine C produces 40%. Then we have

$P(D|A) = 0.02$, $P(A) = 0.35$;

$P(D|B) = 0.01$, $P(B) = 0.25$;

$P(D|C) = 0.03$, $P(C) = 0.40$.

If one spring is selected at random from the total springs produced in a day, the probability that it is defective equals

$P(D) = \sum_{X \in \{A,B,C\}} P(D|X)P(X) = 215/10000$

If the selected spring is defective, the conditional probability that it was produced by machine A, B, *or* C can be calculated by

$P(A|D) = P(D|A)P(A)/P(D) = 70/215$

$P(B|D) = P(D|B)P(B)/P(D) = 25/215$

$P(C|D) = P(D|C)P(C)/P(D) = 120/215$

Example 2: Bowl B_1 contains 2 white chips, bowl B_2 contains 2 red chips, bowl B_3 contains 2 white and 2 red chips, and bowl B_4 contains 3 white chips and 1 red chip. The probabilities of selecting bowl B_1, B_2, B_3, or B_4 are 1/2, 1/4, 1/8, and 1/8, respectively. A bowl is selected using these probabilities and a chip is then drawn at random. Find

(a) $P(W)$, the probability of drawing a white chip.

(b) $P(B_1|W)$, the conditional probability that bowl B_1 had been selected, given that a white chip was drawn.

(c) $P(B_3|W)$, the conditional probability that bowl B_1

Example x: Bowl C contains 6 red chips and 4 blue chips. Five of these chips are selected at random and without replacement and put in bowl D, which was oroginally empty. One chip is then drawn at random from bowl D. Given that this chip is *blue*, find the conditionally probability that *2 red chips* and *3 blue chips* were transferred from bowl C to bowl D.

Scatter Plots, Least Squares, And Correlation

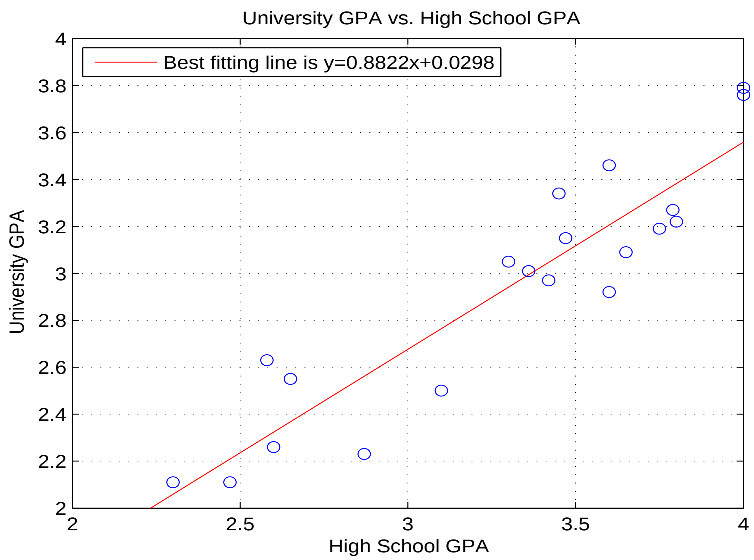
The respective high school and college GPAs for 20 college seniors as ordered pairs (x, y) are

(3.75, 3.19)	(3.45, 3.34)	(2.87, 2.23)	(3.60, 3.46)	(3.42, 2.97)
(4.00, 3.79)	(2.65, 2.55)	(3.10, 2.50)	(3.47, 3.15)	(2.60, 2.26)
(4.00, 3.76)	(2.30, 2.11)	(2.47, 2.11)	(3.36, 3.01)	(3.60, 2.92)
(3.65, 3.09)	(3.30, 3.05)	(2.58, 2.63)	(3.80, 3.22)	(3.79, 3.27)

(a) Verify that $\bar{x} = 3.2880$, $\bar{y} = 2.9305$, $s_x^2 = 0.283$, $s_y^2 = 0.260$, and $r = 0.92$.

(b) The equation of best fitting line is $y = 0.8822x + 0.0298$.

(c) Plot the 20 points and the best fitting line on the same graph.



Suppose that we perform a random experiment n times, obtaining n observed values of the random variables, say, $x_1, x_2, \cdots, x_n$. Often, the collection is referred to as **a sample**.

mean of the empirical distribution $\quad \mu = \sum_{i=1}^{n} x_i \left(\frac{1}{n}\right) = \frac{1}{n}\sum_{i=1}^{n} x_i$

sample mean $\quad \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

variance of the empirical distribution $\quad \sigma^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 \left(\frac{1}{n}\right) = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2$

sample variance $\quad s^2 = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \bar{x})^2$