Partial Solutions for h3/2014S

(1)(a) $E(X) = \int_0^\pi \frac{x}{2}\sin(x)dx = \pi/2$, $Var(X) = \frac{\pi^2}{4} - 2$

(1)(b)(c)

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% Script file: h3p1.m - Problem 1(b)(c) of H3
% Plots of f(x)=(1/2)sin(x) and F(x)=(1/2)[1-cos(x)],  0<=x<=\pi
%
X=0:(pi/32):pi;
f=0.5*sin(X);
F=0.5*(1-cos(X));
plot(X,f,'b-',X,F,'r-');
axis([0,pi, 0,1]);
legend('f(x)=(1/2)sin(x)','F(x)=(1/2)[1-cos(x)]',2);
title('f(x)=(1/2)sin(x) and F(x)=(1/2)[1-cos(x)] for 0 <= x <= \pi')
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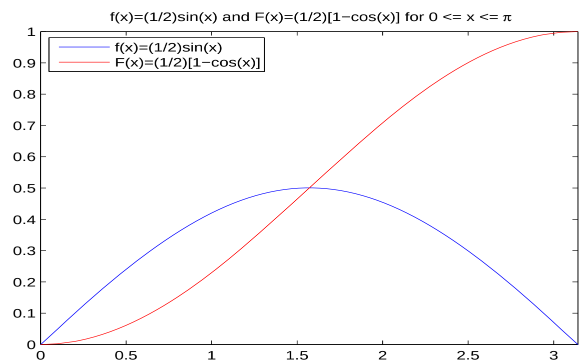

(2) Let X have the p.d.f. $f(x) = \theta x^{\theta-1}$, $0 < x < 1$, $0 < \theta < \infty$, and let $Y = -2\theta ln X$.

(a) $P(Y \le y) = P(X \ge e^{-y/2\theta}) = \int_{e^{-y/2\theta}}^1 \theta x^{\theta-1}dx = 1 - e^{-y/2}$, $y > 0$, then $f(y) = \frac{1}{2}e^{-y/2}$, $y > 0$, thus, Y has an exponential distribution with mean 2.

(b) $M_Y(t) = \frac{1}{1-2t}$, $t < \frac{1}{2}$.

(3) Write down the following probability density functions and *derive* their moment generating functions.

(a) *Exponential distribution with variance 4* ($\phi(t) = \frac{1}{1-\mathbf{2}t}$).

(b) *Normal distribution with mean 3, variance 4* ($\phi(t) = e^{3t+2t^2}$).

(c) χ^2 *distribution with the degrees of freedom 12* ($\phi(t) = \frac{1}{(1-2t)^6}$).

(4)(5)(6)

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% (4) Exponential Distribution
%
subplot(2,2,1)
X=0.1:0.1:12;
Ya=exppdf(X,1); Yb=exppdf(X,2); Yc=exppdf(X,4); Yd=exppdf(X,7);
plot(X,Ya,'r-',X,Yb,'g-',X,Yc,'b-',X,Yd,'m-'); %axis([0,12, 0,0.3])
legend('Exp(1)','Exp(2)','Exp(4)','Exp(7)')
title('(4) Exponential(\theta), \theta=1,2,4,7')
%
% (5) Chi-Square Distribution
%
subplot(2,2,2)
X=0.1:0.1:12;
Y1=chi2pdf(X,1); Y2=chi2pdf(X,2); Y4=chi2pdf(X,4); Y7=chi2pdf(X,7);
plot(X,Y1,'r-',X,Y2,'g-',X,Y4,'b-',X,Y7,'m-'); %axis([0,12, 0,0.3])
legend('\chi^2(1)','\chi^2(2)','\chi^2(4)','\chi^2(7)')
title('(5) \chi^2(r), r=1,2,4,7')
%
% (6) Normal Distribution
%
subplot(2,2,3)
X7=-6:0.2:6;   u=0;  s1=1;  s2=2;  s3=2.5;  s4=3;
Y7a=normpdf(X7,u,s1); Y7b=normpdf(X7,u,s2); Y7c=normpdf(X7,u,s3);
Y7d=normpdf(X7,u,s4);
plot(X7,Y7a,'r-',X7,Y7b,'g-',X7,Y7c,'b-',X7,Y7d,'m-');axis([-6,6, 0,0.42])
legend('N(0,1)','N(0,4)','N(0,6.25)','N(0,9)')
title('(6) Normal Distribution:  N(u,s^2)')
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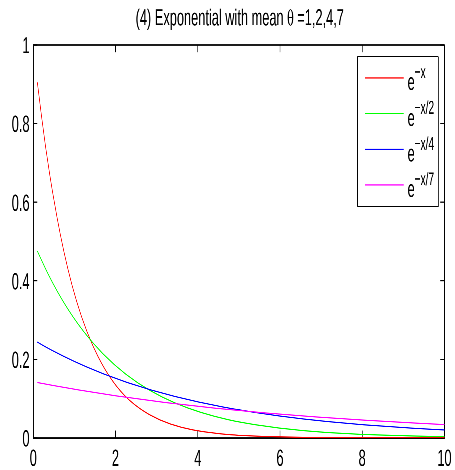
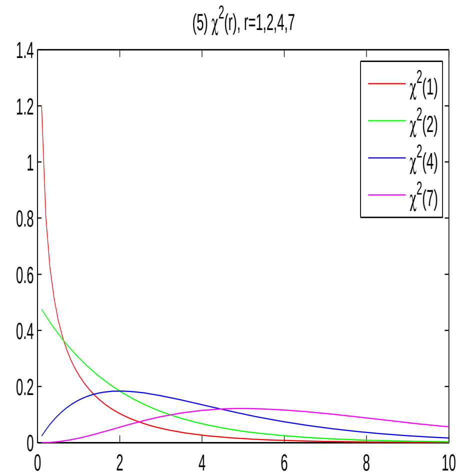
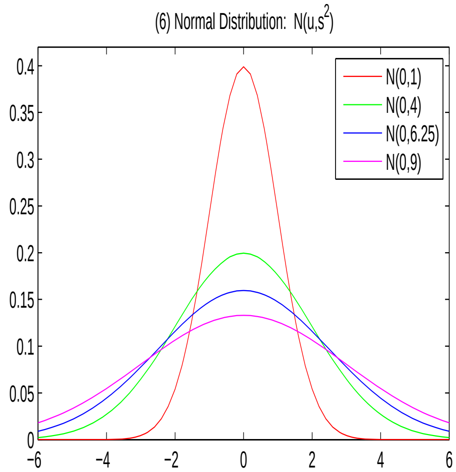

(4) Exponential with mean θ =1,2,4,7
e^{-x}
$e^{-x/2}$
$e^{-x/4}$
$e^{-x/7}$
0
0.2
0.4
0.6
0.8
1
0
2
4
6
8
10


(5) $\chi^2(r)$, r=1,2,4,7
$\chi^2(1)$
$\chi^2(2)$
$\chi^2(4)$
$\chi^2(7)$
0
0.2
0.4
0.6
0.8
1
1.2
1.4
0
2
4
6
8
10


(6) Normal Distribution: N(u,s^2)
N(0,1)
N(0,4)
N(0,6.25)
N(0,9)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
−6
−4
−2
0
2
4
6

(7) Let X have a logistic distribution with the p.d.f.

$$f(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \quad -\infty < x < \infty$$

Proof:

$$\begin{aligned} F(y) &= P(Y \le y) \\ &= P(\tfrac{1}{1+e^{-X}} \le y) \\ &= P(X \le \ln(\tfrac{y}{1-y})) \\ &= \int_{-\infty}^{\ln(\frac{y}{1-y})} \tfrac{e^{-x}}{(1+e^{-x})^2} dx \\ &= y \end{aligned}$$

for $0 < y < 1$, then $f(y) = F'(y) = 1 \;\; for \; y \in (0,1)$, hence $Y = \frac{1}{1+e^{-X}}$ has a U(0,1).

(8)(a) $P(10 < X < 30) = e^{-0.5} - e^{-1.5} = 0.6065 - 0.2231 = 0.3834$

(8)(b) $P(X > 30) = e^{-1.5} = 0.2231$

(8)(c) $P(X > 40|X > 10) = e^{-1.5} = 0.2231$

(9) The p.d.f. of time X to failure of an electronic component is

$$f(x) = \frac{2x}{10^6} e^{-(x/1000)^2}, \quad 0 < x < \infty$$

(a) $P(X > 2000) = e^{-4} \approx 0.0183$.

(b) $q_3 = \pi_{0.75} \approx 1177.4$.

(c) $\pi_{0.10} \approx 324.6, \quad \pi_{0.60} \approx 957.2$.

(10)(a) $f(x) = \frac{2}{\sqrt{2\pi}} e^{-2(x-3)^2}, \quad -\infty < x < \infty.$

(10)(b) $F(x) = P(Z \leq x) = P(2(X-3) \leq x) = P(X \leq \frac{x}{2}+3) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi(0.5)}} e^{-(t-3)^2/2(0.25)} dt = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$

(10)(c) $M_Z(t) = e^{t^2/2}.$

(10)(d) $F(x) = P(Z^2 \leq x) = P(-\sqrt{x} \leq Z \leq \sqrt{x}) = \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \int_{0}^{\sqrt{x}} \frac{2}{\sqrt{2\pi}} e^{-z^2/2} dz,$ thus $f(x) = F'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty.$

(10)(e) $M_W(t) = \frac{1}{\sqrt{1-2t}}.$

(10)(f) $M_V(t) = \exp(t^2).$

(10)(g) $V \sim N(0, 2).$

(10)(h) $f_V(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}, \quad -\infty < x < \infty.$