Solutions for Exam 2 of CS3332(01), Spring 2015

10:10-11:50, Wednesday, June 17, 2015

$Name:$ ____________ $SN:$ ___________ $Gn:$ _____ $Index:$ _____

(30pts) 1. Fill the following blanks (no partial credits).

(a) Let $\{X_i \sim b(5, 0.8),\ 1 \le i \le 10\}$ be a random sample and let $X = \sum_{i=1}^{10} X_i$. Then

the p.m.f. of X, $f_X(x) =$ $\underline{C_x^{50}(0.8)^x(0.2)^{50-x},\ \ 0 \le x \le 50}$

the moment-generating function $M_X(t) =$ $\underline{(0.2 + 0.8e^t)^{50}}$

(b) Let $\{Y_i,\ 1 \le i \le 10\}$ be a random sample of Poisson distribution with variance $Var(Y_1) = 0.8$ and define $Y = \sum_{i=1}^{10} Y_i$. Then

the p.m.f. of Y, $f_Y(y) =$ $\underline{\frac{e^{-8}8^y}{y!},\ y = 0, 1, 2, \cdots, \infty}$

the moment-generating function $M_Y(t) =$ $\underline{e^{8(e^t-1)}}$

(c) Define $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$, then

$\Gamma(5) = \underline{24}$, $\Gamma(7) = \underline{720}$

$\Gamma(\frac{1}{2}) = \underline{\sqrt{\pi}}$, $\Gamma(\frac{5}{2}) = \underline{\frac{3}{4}\sqrt{\pi}}$

(d) Let X be a random variable having an *exponential* distribution with the variance $Var(X) = 9$, then

the p.d.f. of X, $f_X(x) =$ $\underline{\frac{1}{3}e^{-x/3},\ x > 0}$

the moment-generating function $M_X(t) =$ $\underline{\frac{1}{1-3t},\ t < \frac{1}{3}}$

(e) Let $X \sim \chi^2(8)$, that is, X has a *Chi-square* distribution with the degrees of freedom 8, then

the p.d.f. of X, $f_X(x) =$ $\underline{\frac{1}{\Gamma(4)2^4}x^3e^{-x/2},\ x > 0}$

the moment-generating function $M_X(t) =$ $\underline{\frac{1}{(1-2t)^4},\ t < \frac{1}{2}}$ and the variance $Var(X) =$ $\underline{16}$

(30pts) 2. Fill the following blanks (no partial credits).

(a) Let $\{X_i,\ 1 \le i \le 9\}$ be a random sample of size 9 from $N(2,1)$. Define $\overline{X} = \frac{1}{9}\sum_{i=1}^{9} X_i$ and $Y = \sum_{i=1}^{9}(X_i - 2)^2$. Then

the moment-generating function $M_{\overline{X}}(t) =$ $\underline{e^{2t+(t^2/18)}}$

the moment-generating function $M_Y(t) =$ $\underline{\frac{1}{(1-2t)^{9/2}}}$

(b) Let $\{Z_i \sim N(0,1),\ 1 \le i \le 10\}$ be a random sample of size 10. Define $V = \sum_{i=1}^{10} Z_i$ and $W = \sum_{i=1}^{10} Z_i^2$. Then

the moment-generating function $M_V(t) =$ $\underline{e^{5t^2}}$

the moment-generating function $M_W(t) =$ $\underline{\frac{1}{(1-2t)^5}}$

(c) Let $\{X_1, X_2, \cdots, X_n\}$ be a random sample of size n from the *exponential distribution* with $E(X_n) = 3$. Define $Y = \sum_{i=1}^{n} X_i$. Then

the p.d.f. of Y, $f_Y(y) =$ $\underline{\frac{1}{\Gamma(n)3^n} y^{n-1} e^{-y/3},\ y > 0}$

the moment-generating function of Y, $\phi_Y(t) =$ $\underline{\frac{1}{(1-3t)^n}}$

(d) Let $Z \sim N(0,1)$. Define $Y = 3Z + 2$, then

the p.d.f. of Y, $f_Y(y) =$ $\underline{\frac{1}{\sqrt{18\pi}} exp[-\frac{(y-2)^2}{18}],\ \ -\infty < y < \infty}$

the moment-generating function of Y, $\phi_Y(t) =$ $\underline{e^{2t+\frac{9t^2}{2}}}$

(e) Let $\{X_i \sim \chi^2(1),\ 1 \le i \le n\}$ be a random sample and define $W_n = (\sum_{i=1}^{n} X_i - n)/\sqrt{2n}$. According to the Central Limit Theorem,

the limiting distribution function of W_n $limit_{n\to\infty} P(W_n \le w) =$ $\underline{\int_{-\infty}^{w} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \quad = \ \Phi(w)}$

the limiting moment-generating function is $limit_{n\to\infty} M_{W_n}(t) =$ $\underline{e^{t^2/2}}$

(10pts) 3. Let the r.v. X have the p.d.f. $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$, where $\alpha, \beta > 0$ are known positive integers.

Find $E[X]$ and $Var[X]$.

Hint: $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ and $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

See Lecture Notes

(10 pts) 4. Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be the order statistics of a random sample $\{X_1, X_2, \cdots, X_n\}$ from the uniform distribution U(0,1).

(a) Find the probability density function of $X_{(1)}$.

(b) Use the results of **(a)** to find $E[X_{(1)}]$.

(c) Find the probability density function of $X_{(n)}$.

(d) Use the result of **(c)** to find $E[X_{(n)}]$.

Hint: Let $\{X_1, X_2, \cdots, X_n\}$ be a random sample from $U(0, 1)$, then $X_{(1)} = min\{X_1, X_2, \cdots, X_n\}$ and $X_{(n)} = max\{X_1, X_2, \cdots, X_n\}$.

See Lecture Notes

(10pts) 5. Let X have the p.d.f. $f(x) = \beta x^{\beta-1}, \quad 0 < x < 1$ for a given $\beta > 0$. Define $Y = -3\beta ln(X)$.

(a) Compute the distribution function of Y, $P(Y \leq y),\ for\ 0 < y < \infty$.

(b) Find the moment-generating function $M_Y(t)$.

(c) Find $E(Y)$ and $Var(Y)$.

See Lecture Notes

(10pts) 6. In probability theory and statistics, *skewness* is a measure of the asymmetry of the probability distribution of a real-valued random variable about its mean. Similarly, *kurtosis* is a measure of the "peakness" of the probability distribution of a real-valued random variable. Let X be a continuous random variable having $E(X) = \mu$ and $Var(X) = \sigma^2$. Define

$$\begin{aligned} Skew(X) &= E[(X-\mu)^3]/\sigma^3 \\ Kurtosis(X) &= E[(X-\mu)^4]/\sigma^4 \end{aligned}$$

Let $Z \sim N(0,1)$ and $W \sim N(1,4)$. Show your processes to solve (**a** $\sim$ **d**).

(a) Compute $Skew(Z)$.

(b) Compute $Kurtosis(Z)$.

(c) Compute $Skew(W)$.

(d) Compute $Kurtosis(W)$.

Solution:

(a) $Skew(Z) = E[(Z-0)^3]/1^3 = E(Z^3) = M^{(3)}(0) = 0$.

(b) $Kurtosis(Z) = E[(Z-0)^4]/1^4 = E(Z^4) = M^{(4)}(0) = 3$.

(c) $Skew(W) = 0$ since $\frac{W-E(W)}{\sqrt{Var(W)}} \sim N(0,1)$.

(d) $Kurtosis(W) = 3$ since $\frac{W-E(W)}{\sqrt{Var(W)}} \sim N(0,1)$.