(30pts) 1. Choose the best (unique) solution for each of the following problems.

(a) Let \( X \) be a gamma distribution with p.d.f. \( f(x) = \frac{1}{16}x^2e^{-x/2}, \ 0 \leq x \leq \infty \), then the moment-generating function is

\begin{align*}
(1) & \quad \frac{1}{(1-2t)^2},
(2) & \quad \frac{1}{(1-3t)^2},
(3) & \quad \frac{1}{1-2t},
(4) & \quad \frac{1}{1-3t},
(5) & \quad \text{none}
\end{align*}

(b) Let \( X \) have a density function \( f(x) = \frac{1}{2}e^{-x/2}, \ x \geq 0 \), then the variance of \( X \) is

\begin{align*}
(1) & \quad 2,
(2) & \quad 4,
(3) & \quad \frac{1}{2},
(4) & \quad \frac{1}{4},
(5) & \quad \text{none}
\end{align*}

(c) Let \( X \) have a density function \( f(x) = \frac{1}{2}e^{-x/2}, \ x \geq 0 \), then the median of \( X \) is

\begin{align*}
(1) & \quad e^{-1},
(2) & \quad e^{-2},
(3) & \quad 2\ln 2,
(4) & \quad (\ln 2)/2,
(5) & \quad \text{none}
\end{align*}

(d) Let \( X \) be a gamma distribution with p.d.f. \( f(x) = \frac{1}{16}x^2e^{-x/2}, \ 0 \leq x \leq \infty \), then the variance of \( X \) is

\begin{align*}
(1) & \quad 3,
(2) & \quad 6,
(3) & \quad 12,
(4) & \quad 16,
(5) & \quad \text{none}
\end{align*}

(e) Let \( X \) be a \( \chi^2 \) distribution whose moment-generating function \( \phi(t) = \frac{1}{(1-2t)^2}, \ t < \frac{1}{2} \), then the variance \( \text{Var}(X) \) is

\begin{align*}
(1) & \quad 1,
(2) & \quad 2,
(3) & \quad 4,
(4) & \quad 8,
(5) & \quad \text{none}
\end{align*}

(f) Let \( X \) be a \( \chi^2 \) distribution whose moment-generating function \( \phi(t) = \frac{1}{(1-2t)^2}, \ t < \frac{1}{2} \), then the first quartile of \( X \) is

\begin{align*}
(1) & \quad 2 \ln\left(\frac{4}{3}\right),
(2) & \quad 2 \ln\left(\frac{2}{3}\right),
(3) & \quad 0.25,
(4) & \quad 0.75,
(5) & \quad \text{none}
\end{align*}

(g) Let \( X \sim N(3, 4) \), then the moment-generating function of \( X \) is

\begin{align*}
(1) & \quad e^{3t+4t^2},
(2) & \quad e^{3t+2t^2},
(3) & \quad \frac{1}{1-3t^2},
(4) & \quad \frac{1}{3-4t^2},
(5) & \quad \text{none}
\end{align*}

(h) Let \( X \sim N(3, 4) \), then the median of \( X \) is

\begin{align*}
(1) & \quad 1,
(2) & \quad 2,
(3) & \quad 3,
(4) & \quad 4,
(5) & \quad \text{none}
\end{align*}

(i) Let \( Z \sim N(0, 1) \) and define \( X = 2Z + 3 \), then \( \text{Var}(X) \) equals

\begin{align*}
(1) & \quad 1,
(2) & \quad 2,
(3) & \quad 3,
(4) & \quad 4,
(5) & \quad \text{none}
\end{align*}

(j) Let \( X \sim N(3, 4) \) and define \( Y = (X - 3)/2 \), then \( \text{Var}(Y^2) \) equals

\begin{align*}
(1) & \quad 1,
(2) & \quad 2,
(3) & \quad 3,
(4) & \quad 4,
(5) & \quad \text{none}
\end{align*}
(40pts) 2. Fill the following blanks.

(a) Let $A$ and $B$ be independent events with $P(A) = 0.7$ and $P(B) = 0.2$, then
\[ P(A \cap B) = \boxed{0.14} \]
\[ P(A \cup B) = \boxed{0.76} \]

(b) Let $X \sim b(50, 0.6)$, then
\[ E(X) = \boxed{30} \]
\[ Var(X) = \boxed{12} \]
\[ M(t) = (0.4 + 0.6e^t)^{50} \]

(c) If the moment-generating function of $X$ is $M(t) = \frac{2}{5}e^t + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t}$, then
\[ E(X) = \boxed{2} \]
\[ Var(X) = \boxed{\frac{4}{5}} \]

(d) A random variable $X$ has p.d.f. $f(x) = \frac{x+1}{2}$, $-1 < x < 1$. Then
the median of $X = -1 + \sqrt{2}$

(e) Let $X \sim N(3, 4)$, then
the p.d.f. of $X$ is $f(x) = \frac{1}{\sqrt{8\pi}}e^{-(x-3)^2/8}$
the moment-generating function $\phi_X(t) = e^{3t+2t^2}$

(f) Let a random variable $X$ have the Poisson distribution with variance 4.
\[ E(X) = \boxed{4} \]
\[ M(t) = e^{4(e^t-1)} \]

(g) Let a random variable $X$ have the geometric distribution with $E(X) = 4$. Then
\[ Var(X) = \boxed{12} \]
\[ M(t) = \frac{e^t}{4-3e^t} \]

(h) Let a random variable $X$ have an exponential distribution with $Var(X) = 4$. Then
\[ E(X) = \boxed{2} \]
The moment-generating function $\phi_X(t) = \frac{1}{1-2t}$
(10pts) 3. A box contains four marbles numbered 1 through 4. The marbles are selected one at a time without replacement. A match occurs if marble numbered \( k \) is the \( k \)th marble selected. Let the event \( A_i \), denote a match on the \( i \)th draw, \( i=1,2,3,4 \).

(a) Find \( P(A_i) \), for \( i=1,2,3,4 \).
(b) Find \( P(A_i \cap A_j) \), where \( 1 \leq i < j \leq 4 \).
(c) Find \( P(A_i \cap A_j \cap A_k) \), where \( 1 \leq i < j < k \leq 4 \).
(d) Find \( P(A_1 \cup A_2 \cup A_3 \cup A_4) \).

See Lecture Notes

(10pts) 4. Let the random variable \( X \) have a negative binomial distribution whose probability mass function (p.m.f) is

\[
 f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad \forall \ x = r, \ r+1, \ r+2, \ \cdots
\]

(a) Show your procedure to find the moment-generating function \( M(t) \).
(b) Show your procedure to find the expectation \( E(X) \).
(c) Compute the variance \( Var(X) \).

Hint: \( (1-\omega)^{-r} = \sum_{k=0}^{\infty} \binom{k+r-1}{r-1} \omega^k \)

See Lecture Notes

(10pts) 5. Write Simple Matlab Codes to plot each of the following probability mass functions with a given random variable \( X \).

(a) \( X \sim b(10,0.8) \).
(b) \( X \) has a geometric distribution with the expectation \( E(X) = 4 \).
(c) \( X \) has a Poisson distribution with variance 4.

See Homework Assignment