(20pts) 1. For the following statements, mark a ⃝ if it is true, and mark a × otherwise.

(a) ×, (b) ×, (c) ×, (d) ×, (e) ×, (f) ⃝, (g) ⃝, (h) ⃝, (i) ⃝, (j) ⃝, (k) ⃝, (l) ⃝, (m) ⃝

(a) Let $X_i \sim b(5, \frac{1}{2})$, $1 \leq i \leq 8$, be a random sample of size 8. Then $\sum_{i=1}^{8} X_i \sim b(8, \frac{1}{2})$.

(b) Let $X_1, X_2, X_3$ be mutually independent r.v.'s with Poisson distributions having mean 1, 2, 3, respectively. Then $\sum_{i=1}^{3} X_i$ has a Poisson distribution with variance 14.

(c) Define $\Gamma(x) = \int_{0}^{\infty} e^{-t} t^{x-1} dt$, then $\Gamma(4) = 4! = 24$.

(d) Define $\Gamma(x) = \int_{0}^{\infty} e^{-t} t^{x-1} dt$, then $\int_{0}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{4}$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

(e) Let $Y \sim \chi^2(3)$, then the moment-generating function of $Y$ is $\phi_Y(t) = \frac{1}{(1 - 2t)^3}$.

(f) Let $\{X_i, 1 \leq i \leq 5\}$ be a random sample of size 5 from $N(2, 1)$. Define $Y = \sum_{i=1}^{5} (X_i - 2)^2$, then $Y \sim \chi^2(5)$.

(g) Let $\{Z_i, 1 \leq i \leq n\}$ be a random sample of size $n$ from the standard normal distribution. Then $W = \sum_{i=1}^{n} Z_i^2 \sim \chi^2(n)$ and $Var(W) = 2n$.

(h) Let $\{Z_i, 1 \leq i \leq n\}$ be a random sample of size $n$ from the standard normal distribution. Then $\bar{Z} = \frac{1}{n} \sum_{i=1}^{n} Z_i \sim N(0, \frac{1}{n})$.

(i) Let $X_1, X_2, \cdots, X_n$ be a random sample of size $n$ from the Poisson distribution with mean 4 and define $Y = \sum_{j=1}^{n} X_j$. According to the central limit theorem, $\frac{Y - 4n}{\sqrt{4n}}$ has a limiting distribution $N(0, 1)$.

(j) Let $Y \sim \chi^2(2n)$. According to the central limit theorem, $\frac{Y - 2n}{\sqrt{4n}}$ has a limiting distribution $N(0, 1)$.

(k) Let $\{X_i, 1 \leq i \leq n\}$ be a random sample of size $n$, from an exponential distribution with variance 4. Let $Y = \sum_{i=1}^{n} X_i$. According to the central limit theorem, $\frac{Y - 2n}{\sqrt{4n}}$ has a limiting distribution $N(0, 1)$.
(40pts) 2. Fill the following blanks.

(a) Let $X$ have the binomial distribution with the expectation $E(X) = 40$ and variance $Var(X) = 8$, then

the p.m.f. of $X$, $f_X(x) = \binom{50}{x}(0.8)^x(0.2)^{50-x}, \ 0 \leq x \leq 50$

the moment-generating function $M_X(t) = (0.2 + 0.8e^t)^{50}$

(b) Let $X$ be a random variable having geometric distribution with the expectation $E(X) = \frac{4}{3}$, then

the p.m.f. of $X$, $f_X(x) = \frac{3}{4^x}, \ x = 1, 2, \ldots$

the moment-generating function $M_X(t) = \frac{3e^t}{1-e^t}$

the variance $Var(X) = \frac{4}{9}$

(c) Let $Y$ be a random variable having Poisson distribution with the variance $Var(Y) = 4$, then

the p.m.f. of $Y$, $f_Y(y) = \frac{e^{-4}4^y}{y!}, \ y = 0, 1, 2, \ldots$

the moment-generating function $M_Y(t) = e^{4(e^t-1)}$

(d) Let $X$ be a random variable having an exponential distribution with the variance $Var(X) = 4$, then

the p.d.f. of $X$, $f_X(x) = \frac{1}{2}e^{-x/2}, \ 0 < x < \infty$

the moment-generating function $M_X(t) = \frac{1}{1-2t}$

(e) Let $X \sim \chi^2(4)$, that is, $X$ has a Chi-square distribution with the degrees of freedom 4, then

the p.d.f. of $X$, $f_X(x) = \frac{1}{11(4/2)^{2/2}}x^{(4/2)-1}e^{-x/2}, \ x > 0$

the moment-generating function $M_X(t) = \frac{1}{(1-2t)^{2/4}}$

the variance $Var(X) = 8$
(f) Let $X \sim N(3, 4)$, that is, $X$ has a normal distribution with the expectation $E(X) = 3$ and variance $Var(X) = 4$, then

the p.d.f. of $X$, $f_X(x) = \frac{1}{\sqrt{8\pi}} e^{-(x-3)^2/8}, \; -\infty < x < \infty$.

the moment-generating function $M_X(t) = e^{3t+2t^2}$.

(g) Let $X_i \sim b(10, 0.8)$ for $1 \leq i \leq 10$ be a random sample from the binomial distribution. Define $Y = \sum_{i=1}^{10} X_i$, then

the p.d.f. of $X$, $f_Y(y) = C(100, y)(0.2)^{100-y}, \; 0 \leq y \leq 100$.

the moment-generating function of $Y$, $M_Y(t) = (0.2 + 0.8e^t)^{100}$.

(h) Let $\{X_1, X_2, \cdots, X_n\}$ be a random sample of size $n$ from the exponential distribution with $E(X_1) = 2$. Define $Y = \sum_{i=1}^{n} X_i$. Then

the p.d.f. of $X_n$, $f_{X_n}(x) = \frac{1}{2} e^{-x/2}, \; 0 < x < \infty$.

the p.d.f. of $Y$, $f_Y(y) = \frac{1}{1(1)^2} y^{n-1} e^{-y/2}, \; y > 0$.

the moment-generating function of $Y$, $\phi_Y(t) = \frac{1}{(1-2t)^n}$.

(i) Let $Z \sim N(0, 1)$. Define $Y = 4Z + 3$, then

the p.d.f. of $Y$, $f_Y(y) = \frac{1}{\sqrt{32\pi}} e^{-(y-3)^2/32}, \; \infty < y < \infty$.

the moment-generating function of $Y$, $\phi_Y(t) = e^{3t+8t^2}$.

(j) The joint probability density function for two continuous random variables $X$ and $Y$ is given as follows.

\[ f(x, y) = \begin{cases} 
\beta xy & 0 < x < 2 \text{ and } 0 < y < 4 \\
0 & \text{elsewhere}
\end{cases} \]

(a) The constant $\beta$ to make it a joint p.d.f. is $\beta = \frac{1}{16}$.

(b) The marginal probability density function $f_Y(y) = \frac{y}{8}, \; 0 < y < 4$.

(c) The conditional probability density function $f(x|y) = \frac{x}{2}, \; 0 < x < 2$. 
(10pts) 3. Let the r.v. X have the p.d.f. \( f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1 - x)^{\beta-1}, \) 0 < \( x < 1, \) where \( \alpha, \beta > 0 \) are known parameters.

Find \( E[X] \) and \( Var[X]. \)

**Hint:** \( \Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt \) and \( \int_0^1 x^{\alpha-1}(1 - x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \)

S3:

\[
E[X] = \int_0^\infty x f(x)dx \\
= \int_0^\infty \left[ \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha}(1 - x)^{\beta-1} \right] dx \\
= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \\
= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\alpha\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)} \\
= \frac{\alpha}{\alpha+\beta}
\]

\[
Var[X] = E[X^2] - (E[X])^2 \\
= \int_0^\infty \left[ \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha+1}(1 - x)^{\beta-1} \right] dx - (E[X])^2 \\
= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} - (E[X])^2 \\
= \frac{\alpha^2}{(\alpha+\beta)^2(\alpha+\beta+1)}
\]

(10pts) 4. Let a r.v. X have the probability density function \( f(x) = \frac{\pi}{2} \cos(\pi x), -\frac{1}{2} \leq x \leq \frac{1}{2}. \)

(a) Sketch the graph of the p.d.f. of X.
(b) Find the expectation \( E(X). \)
(c) Find the variance \( Var(X). \)

(a) \( X=-0.5:0.05:0.5; \ Y=(\pi/2)*\cos(\pi*x); \ plot(X,Y,'b') \)
(b) \( E(X) = 0. \)
(c) \( Var(X) = \frac{1}{4} - \frac{2}{\pi^2} \approx 0.0474. \)
(10pts) 5. Let $X$ have the p.d.f. $f(x) = xe^{-x^2/2}$, $0 < x < \infty$. Show that $Y = X^2$ has an exponential distribution with $E(Y) = 2$.

(a) Proof:

$$F(y) = P(Y \leq y) = P(0 \leq X \leq \sqrt{y})$$

$$= \int_0^{\sqrt{y}} xe^{-x^2/2} dx = 1 - e^{-y/2}$$

Thus, $f(y) = F'(y) = \frac{1}{2}e^{-y/2}$, $y > 0$, therefore, $Y \sim \text{Exp}(2)$.

(b) $X=0:0.1:4; \ Y=X.*\exp(-X.^2/2); \ \text{plot}(X,Y,'b-');$

(10 pts) 6. Let $W_1 < W_2 < \cdots < W_n$ be the order statistics of $X_1, X_2, \ldots, X_n$, a random sample of size $n$, from the uniform distribution $U(0,1)$.

(a) Find the probability density function of $W_1$.

(b) Find the probability density function of $W_n$.

(c) Use the result of (a) to find $E(W_1)$.

(d) Use the result of (b) to find $E(W_n)$.

Hint: Let $\{X_1, X_2, \ldots, X_n\}$ be a random sample from $U(0, 1)$, then $W_1 = \min\{X_1, X_2, \ldots, X_n\}$ and $W_n = \max\{X_1, X_2, \ldots, X_n\}$.

S6(a) $P(W_1 \leq x) = 1 - P(W_1 > x) = 1 - \Pi_{i=1}^{n} P(X_i > x) = 1 - (1 - x)^n$. Then $f_{W_1}(x) = n(1 - x)^{n-1}$, $0 < x < 1$.

S6(b) $P(W_n \leq y) = \Pi_{i=1}^{n} P(X_i \leq y) = y^n$. Then $f_{W_n}(y) = ny^{n-1}$, $0 < y < 1$.

S6(c) $E(W_1) = \int_0^1 x \cdot n(1 - x)^{n-1} dx = \frac{1}{n+1}$

S6(d) $E(W_n) = \int_0^1 y \cdot ny^{n-1} dy = \frac{n}{n+1}$

S6(*) $E(W_r) = \frac{r}{n+1}$