I. (50 pts) Write Matlab codes to solve the following problems.

(15 pts) 1. Approximate \( \sin(1) \) by using Taylor expansion for \( f(x) = \sin(x) \) with \( x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) about \( x = 0 \) such that the accuracy is within \( 10^{-4} \).

(10 pts) 2. Apply \( LU - decomposition \) with partial pivoting and back substitution to solve the following linear system of equations.

\[
\begin{align*}
x + \frac{1}{2}y + \frac{1}{3}z + \frac{1}{4}w &= 1 \\
\frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z + \frac{1}{5}w &= 1 \\
\frac{1}{3}x + \frac{1}{4}y + \frac{1}{5}z + \frac{1}{6}w &= 1 \\
\frac{1}{4}x + \frac{1}{5}y + \frac{1}{6}z + \frac{1}{7}w &= 1
\end{align*}
\]

(10 pts) 3. Find the characteristic polynomial of the matrix \( A \) given below and compare the roots of the characteristic equation of \( A \) with those obtained from the matlab command \( \text{eig}(A) \).

\[
A = \begin{bmatrix}
-5 & 2 & 1 \\
2 & -5 & 2 \\
1 & 2 & 4
\end{bmatrix}
\]

(15 pts) 4. Let \( X \sim \chi^2(4) \) with the probability density \( f(x) = \frac{1}{4}xe^{-x/2}, \ x \geq 0 \). Find the 75th percentile of \( X \), that is, \( \beta \) such that \( \int_0^\beta f(x)dx = \frac{3}{4} \).
II. (40 pts) Answer the following questions.

(10 pts) 5. Give the nonlinear system of equations with variables \( x = X(1) \), \( y = X(2) \), \( z = X(3) \) that the following Matlab code attempts to solve.

```matlab
n=3;
X=[1; 1; 1];
Nrun=10;
for k=1:Nrun
    F=[X(1)^3-10*X(1)+X(2)-X(3)+3; ...
       X(2)^3+10*X(2)-2*X(1)-2*X(3)-5; ...
       X(1)+X(2)-10*X(3)+2*sin(X(3))+5];
    A=[3*X(1)^2-10, 1, -1; ...
       -2, 3*X(2)+10, -2; ...
       1, 1, -10+2*cos(X(3))];
    s=0.002; % Perturbation to avoid the near singularity
    dX=(A+s*eye(n))\F;
    X=X-dX;
end
format short
[X'; F'] % Output solution and error (ideally F==0)
```

(10 pts) 6. Define \( f(x) = \sqrt{x+2} \) for \( x \in [0, 3] \).

(a) Show that \( f \) is a contractive mapping.

(b) Find the fixed point of \( f \).

(20 pts) 7. Let \( f : [a, b] \to \mathbb{R} \) and given \( 3n + 1 \) distinct points \( a = x_0 < x_1 < \cdots < x_{3n} = b \). In no more than 50 words, describe the difference among a cubic spline interpolant, Bezier curve, and B-spline curve fitting.
III. (60 pts) Answer the following questions.

(20 pts) 8. For \( x \in (-1, 1) \), define the Chebyshev polynomial of degree \( n \) by

\[
T_n(x) = \cos(n \cos^{-1} x) \quad \forall \ n \geq 0
\]

Denote the inner product by \( \langle T_n, T_m \rangle = \int_{-1}^{1} T_n(x)T_m(x)\frac{1}{\sqrt{1-x^2}}dx \)

(a) Write down \( T_n(x) \) in the polynomial format for \( 0 \leq n \leq 4 \).

(b) Show that \( \langle T_n, T_m \rangle = 0 \) if \( n \neq m \).

(c) Compute \( \langle T_0, T_0 \rangle \), and \( \langle T_n, T_n \rangle \) for \( n \geq 1 \).

(d) Find all of the roots of \( T_n(x) = 0 \).

(20 pts) 9. Using the nodes \( x_0 = 2 \), \( x_1 = 2.5 \), and \( x_2 = 5 \) to find a quadratic interpolating polynomial \( P_2(x) \) for \( f(x) = \frac{1}{x^2} \).

(a) Find the Lagrange polynomials \( L_{2,0}(x), L_{2,1}(x) \), and \( L_{2,2}(x) \).

(b) Write \( P_2(x) \) in form of Lagrange polynomials.

(c) Write \( P_2(x) \) in the form of Newton divided difference.

(d) If \( P_2(x) = \alpha_2 x^2 + \alpha_1 x + \alpha_0 \), what are \( \alpha_2, \alpha_1, \alpha_0 \), respectively?

(20 pts) 10. A natural cubic spline \( S \) on \([0,2]\) is defined by

\[
S(x) = \begin{cases} 
S_0(x) &= a + bx + cx^2 + dx^3, &\text{if } 0 \leq x < 1 \\
S_1(x) &= 2 - (x - 1) - 3(x - 1)^2 + (x - 1)^3, &\text{if } 1 \leq x \leq 2
\end{cases}
\]

Show your process to find \( a, b, c, \) and \( d \), respectively.