Test 3

I. (50 pts) *Write Matlab codes to solve the following problems.*

(15 pts) 1. Approximate $sin(1)$ by using Taylor expansion for $f(x) = sin(x)$ with $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ about $x = 0$ such that the accuracy is within 10^{-4}.

(10 pts) 2. Apply $LU - decomposition$ with partial pivoting and back substitution to solve the following linear system of equations.

$$\begin{array}{ccccccccc} x & + & \frac{1}{2}y & + & \frac{1}{3}z & + & \frac{1}{4}w & = & 1 \\ \frac{1}{2}x & + & \frac{1}{3}y & + & \frac{1}{4}z & + & \frac{1}{5}w & = & 1 \\ \frac{1}{3}x & + & \frac{1}{4}y & + & \frac{1}{5}z & + & \frac{1}{6}w & = & 1 \\ \frac{1}{4}x & + & \frac{1}{5}y & + & \frac{1}{6}z & + & \frac{1}{7}w & = & 1 \end{array}$$

(10 pts) 3. Find the characteristic polynomial of the matrix A given below and compare the roots of the characteristic equation of A with those obtained from the matlab command $eig(A)$.

$$A = \begin{bmatrix} -5 & 2 & 1 \\ 2 & -5 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

(15 pts) 4. Let $X \sim \chi^2(4)$ with the probability density $f(x) = \frac{1}{4}xe^{-x/2}$, $x \geq 0$. Find the 75th percentile of X, that is, β such that $\int_0^\beta f(x)dx = \frac{3}{4}$.

II. (40 pts) *Answer the following questions.*

(10 pts) 5. Give the nonlinear system of equations with variables $x = X(1), y = X(2), z = X(3)$ that the following Matlab code attempts to solve.

```
n=3;
X=[1; 1; 1];
Nrun=10;
for k=1:Nrun
  F=[X(1)^3-10*X(1)+X(2)-X(3)+3; ...
     X(2)^3+10*X(2)-2*X(1)-2*X(3)-5; ...
     X(1)+X(2)-10*X(3)+2*sin(X(3))+5];
  A=[3*X(1)^2-10,          1,              -1; ...
               -2,  3*X(2)+10,              -2; ...
                1,          1, -10+2*cos(X(3))];
  s=0.002;  % Perturbation to avoid the near singularity
  dX=(A+s*eye(n))\F;
  X=X-dX;
end
format short
[X'; F']    % Output solution and error (ideally F==0)
```

(10 pts) 6. Define $f(x) = \sqrt{x+2}\ \ for\ x \in [0,3]$.

(a) Show that f is a contractive mapping.

(b) Find the fixed point of f.

(20 pts) 7. Let $f : [a,b] \rightarrow R$ and given $3n+1$ distinct points $a = x_0 < x_1 < \cdots < x_{3n} = b$. In no more than **50** words, discribe the difference among a cubic spline interpolant, Bezier curve, and B-spline curve fitting.

III. (60 pts) *Answer the following questions.*

(20 pts) 8. For $x \in (-1, 1)$, define the Chebyshev polynomial of degree n by

$$T_n(x) = cos(ncos^{-1}x) \ \forall \ n \geq 0$$

Denote the inner product by $\langle T_n, T_m \rangle = \int_{-1}^{1} T_n(x) T_m(x) \frac{1}{\sqrt{1-x^2}} dx$

(a) Write down $T_n(x)$ in the polynomial format for $0 \leq n \leq 4$.

(b) Show that $\langle T_n, T_m \rangle = 0$ if $n \neq m$.

(c) Compute $\langle T_0, T_0 \rangle$, and $\langle T_n, T_n \rangle$ for $n \geq 1$.

(d) Find all of the roots of $T_n(x) = 0$.

(20 pts) 9. Using the nodes $x_0 = 2,\ x_1 = 2.5,\ and\ x_2 = 5$ to find a quadratic interpolating polynomial $P_2(x)$ for $f(x) = \frac{1}{x^2}$.

(a) Find the Lagrange polynomials $L_{2,0}(x), L_{2,1}(x),\ and\ L_{2,2}(x)$.

(b) Write $P_2(x)$ in form of Lagrange polynomials.

(c) Write $P_2(x)$ in the form of Newton divided difference.

(d) If $P_2(x) = \alpha_2 x^2 + \alpha_1 x + \alpha_0$, what are $\alpha_2, \alpha_1, \alpha_0$, respectively?

(20 pts) 10. A natural cubic spline S on [0,2] is defined by

$$S(x) = \begin{cases} S_0(x) &= a + bx + cx^2 + dx^3, & if\ 0 \leq x < 1 \\ S_1(x) &= 2 - (x-1) - 3(x-1)^2 + (x-1)^3, & if\ 1 \leq x \leq 2 \end{cases}$$

Show your process to find $a, b, c,\ and\ d$, respectively.