H5: Eigenvalues/Eigenvectors

♣ **Show your procedures to work out the following problems**.

1. Let $A \in R^{n\times n}$ have eigenvalues $2, 4, \cdots, 2n$. Show that $tr(A) = n(n+1)$ and $det(A) = 2^n \cdot n!$.

2. Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$,

(a) Show that the characteristic polynomial of A is $\lambda^3 - 2\lambda^2 + \lambda$.

(b) Find the eigenvalues and corresponding eigenvectors of matrix A.

(c) Find the eigenspace of each eigenvalue of A.

3. For the matrix B given in problem **2**,

(a) Find the eigenvalues and eigenvectors of B.

(b) Calculate $\|B\|_1$, $\|B\|_\infty$, $\|B\|_2$.

(c) Give a spectrum decomposition of matrix B.

(d*) Give a singular value decomposition (SVD) for matrix B.

(e*) Compute e^B.

4. Verify your solutions for problems **2.(a,b)** and **3.(a,b)** by using Matlab (Show your Matlab codes).

Solutions for H5: Eigenvalues/Eigenvectors

1. $tr(A) = trace(A) = \sum_{k=1}^{n}(2k) = n(n+1)$, $det(A) = \Pi_{k=1}^{n}(2k) = 2^n \cdot n!$

2(a) $det(\lambda I - A) = \lambda^3 - 2\lambda^2 + \lambda$, $\lambda = 0 or 1$ (poly(A)=[1,-2,1,0]).

2(b) [U, D]=eig(A). For $\lambda = 0$, $\mathbf{u}_1 = \frac{1}{\sqrt{3}}[1,1,1]^t$; for $\lambda = 1$, $\mathbf{u} = \frac{\alpha}{\sqrt{2}}[1,0,-1]^t + \frac{\beta}{\sqrt{10}}[0,1,3]^t$, where $\alpha^2 + \beta^2 \neq 0$; we can choose $\mathbf{u}_2 = \frac{1}{\sqrt{2}}[1,0,-1]^t$ and $\mathbf{u}_3 = \frac{1}{\sqrt{10}}[0,1,3]^t$, respectively, corresponding to the eigenvalue 1.

2(c) Eigenspace(0)= $span([1,1,1]^t)$; Eigenspace(1)=$span(\frac{1}{\sqrt{2}}[1,0,-1]^t, \frac{1}{\sqrt{10}}[0,1,3]^t)$.

3(a) $\lambda_1 = -1,\ \mathbf{u}_1 = \frac{1}{\sqrt{2}}[1,1,0]^t, \quad \lambda_2 = -3,\ \mathbf{u}_2 = \frac{1}{\sqrt{2}}[1,-1,0]^t, \quad \lambda_3 = 2,\ \mathbf{u}_3 = [0,0,1]^t,$

3(b) $\|B\|_1 = 3, \quad \|B\|_2 = 3, \quad \|B\|_\infty = 3$

3(c) $B = UDU^t$, where $U = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$, $D = diag[-1,-3,2]$

3(d) $\sigma_1 = 3, \quad \sigma_2 = 2, \quad \sigma_3 = 1$, $B = USV^t$, where $S = diag[3,2,1]$, and $U = [\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_1]$, $V = [-\mathbf{u}_2, \mathbf{u}_3, -\mathbf{u}_1]$,

3(e) $e^B = U diag(e^{-1}, e^{-3}, e^2) U^t$

2(a∼b) By Matlab,

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 poly(A), [U, D]=eig(A);
norm(A,1), norm(A,2), norm(A, inf)
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3(a∼b) By Matlab,

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 poly(B), [U, D]=eig(B);
norm(B,1), norm(B,2), norm(B, inf)
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4(c∼d)
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check if norm(B-U*D*U',2) = 0? ;
[U S V]=svd(B)
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