H4: Orthogonality

1. Given $\mathbf{x} = [1, 1, 1, 1]^t$ and $\mathbf{y} = [8, 2, 2, 0]^t$.

(a) Determine the angle between $\mathbf{x}$ and $\mathbf{y}$.

(b) Find the vector projection $\mathbf{p}$ of $\mathbf{x}$ onto $\mathbf{y}$.

(c) Verify that $(\mathbf{x} - \mathbf{p}) \perp \mathbf{y}$.

2. Compute $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$ and $\|\mathbf{x}\|_\infty$ for each of the following vectors.

$$(a)\ \ \mathbf{x} = [-3, 4, 0]^t, \quad (b)\ \ \mathbf{x} = [-1, -1, 2]^t$$

3. Let $\mathbf{u} = [2, 3, 4]^t$, $\mathbf{v} = [1, \gamma, 1]^t$. Find γ such that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

4. Let $\mathbf{x} = [1, 2, 3]^t$, $\mathbf{y} = [-2, 3, 1]^t$. Find the angle θ between $\mathbf{x}$ and $\mathbf{y}$.

5. Let $\mathbf{x} = [1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots]^t$, $\mathbf{u} = [1, 0, 0, \cdots]^t$. Show that (a) $\|\mathbf{x}\|_2 = \frac{2}{\sqrt{3}}$ and (b) the angle between $\mathbf{x}$ and $\mathbf{u}$ is $\cos^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{6}$.

6. Let $\mathbf{x} \in R^n$ and show that

$$(a)\ \ \|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1, \quad (b)\ \ \|\mathbf{x}\|_1 \leq \sqrt{n}\|\mathbf{x}\|_2, \quad (c)\ \ \|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty$$

7. Sketch the set of points $\mathbf{x} = [x_1, x_2]^t \in R^2$ such that

$$(a)\ \ \|\mathbf{x}\|_2 = 1, \quad (b)\ \ \|\mathbf{x}\|_1 = 1, \quad (c)\ \ \|\mathbf{x}\|_\infty = 1$$

8. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be an orthonormal basis for an inner product vector space V and let $\mathbf{x} = \mathbf{u}_1 + 2\mathbf{u}_2 + 2\mathbf{u}_3$ and $\mathbf{y} = \mathbf{u}_1 + 7\mathbf{u}_2$. Compute

$$(a)\ \langle \mathbf{x}, \mathbf{y} \rangle, \quad (b)\ \|\mathbf{x}\|_2, \|\mathbf{y}\|_2, \quad (c)\ \textit{the angle between } \mathbf{x} \textit{ and } \mathbf{y}$$

9. Find the best least squares fitting line for the data set $\{[-1, 0]^t, [0, 1]^t, [1, 3]^t, [2, 9]^t\}$ and plot your solution associated with the data points.

10. Let $\mathbf{x} = [1, 1, 3, 5]^t$ and let $Q_j \in R^{4\times4}$ be orthogonal for $1 \leq j \leq 4$, find

$$(a)\ \|Q\mathbf{x}\|_2 \ \ \textit{and} \ \ (b)\ \textit{det}(Q_1Q_2Q_3Q_4)$$

Solutions for H4: Orthogonality

1. Since $\mathbf{x} = [1, 1, 1, 1]^t$, $\mathbf{y} = [8, 2, 2, 0]^t$, then $\|\mathbf{x}\|_2 = 2$, $\|\mathbf{y}\|_2 = 6\sqrt{2}$, and $\langle \mathbf{x}, \mathbf{y}\rangle = 12$, then

(a) $\cos\theta = \frac{\langle \mathbf{x},\mathbf{y}\rangle}{\|\mathbf{x}\|_2 \times \|\mathbf{y}\|_2} = \frac{1}{\sqrt{2}}$, then $\theta = \frac{\pi}{4}$.

(b) $\mathbf{p} = \frac{\mathbf{y}\mathbf{y}^t}{\|\mathbf{y}\|_2^2}\mathbf{x} = \frac{1}{3}[4, 1, 1, 0]^t$.

(c) Verify $(\mathbf{x} - \mathbf{p}) \perp \mathbf{y}$. by showing that $\langle \mathbf{x} - \mathbf{p}, \mathbf{y}\rangle = 0$.

2(a) $\|\mathbf{x}\|_1 = 7$, $\|\mathbf{x}\|_2 = 5$, $\|\mathbf{x}\|_\infty = 4$

2(b) $\|\mathbf{x}\|_1 = 4$, $\|\mathbf{x}\|_2 = \sqrt{6}$, $\|\mathbf{x}\|_\infty = 2$

3. $\gamma = -2$

4. $\frac{\pi}{3}$

5. Let $\mathbf{x} = [1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots]^t$, $\mathbf{u} = [1, 0, 0, \cdots]^t$. Show that (a) $\|\mathbf{x}\|_2 = \frac{2}{\sqrt{3}}$ and (b) the angle between $\mathbf{x}$ and $\mathbf{u}$ is $cos^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{6}$.

6. Let $|x_k| = \|\mathbf{x}\|_\infty$, then show that

$$(a)\ \|\mathbf{x}\|_2^2 \leq \|\mathbf{x}\|_1^2 \ \ (b)\ \|\mathbf{x}\|_1^2 \leq n\|\mathbf{x}\|_2^2 \ by Cauchy{-}Schwarz\, inequality \ \ (c)\ \|\mathbf{x}\|_2^2 \leq n\|\mathbf{x}\|_\infty^2$$

7. (a) circle, (b) $|x| + |y| = 1$, (c) unit square boundary.

8. (a) $\langle \mathbf{x}, \mathbf{y}\rangle = 15$, (b) $\|\mathbf{x}\|_2 = 3$, $\|\mathbf{y}\|_2 = 5\sqrt{2}$, (c) *the angle between* $\mathbf{x}$ *and* $\mathbf{y}$ *is* $\frac{\pi}{4}$.

9. The best least squares fitting line for the data set $\{[-1, 0]^t, [0, 1]^t, [1, 3]^t, [2, 9]^t\}$ is $y = 2.9x + 1.8$

10. Let $\mathbf{x} = [1, 1, 3, 5]^t$ and let $Q_j \in R^{4\times4}$ be orthogonal for $1 \leq j \leq 4$, then

$$(a)\ \ \|Q\mathbf{x}\|_2 = 6 \quad and \quad (b)\ \ det(Q_1Q_2Q_3Q_4) = \pm 1$$