H3: Vector Space and Linear Transform

Let $(S,\ \oplus,\ \odot)$ be a set with the operation $\mathbf{x} \oplus \mathbf{y}$ being defined for $\mathbf{x}, \mathbf{y} \in S$, and $\alpha \odot \mathbf{x}$ for $\alpha \in T$ and $\mathbf{x} \in S$. Under the definitions for $(\mathbf{1} \sim \mathbf{3})$, determine if $(S,\ \oplus,\ \odot)$ is a vector space over T.

(◯)(1) Let $S = R^+ = \{r \in R |\ r > 0\}$, $T = R$, define $x \oplus y = xy$ and $k \odot x = x^k$.

(◯)(2) Let $S = \{[1, x]|\ x \in R\}$, $T = R$, define $[1, x] \oplus [1, y] = [1, x + y]$ and $c \odot [1, x] = [1, cx]$.

(×)(3) Let $S = R^2$, $T = R$, define $\mathbf{x} \oplus \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \oplus \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 + 1 \\ x_2 + y_2 + 1 \end{bmatrix}$ and $c \odot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$.

(4) Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$. Show that a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ such that $\mathbf{b} = \sum_{i=1}^{3} c_i \mathbf{v}_i$ has $[c_1, c_2, c_3] = [1, 1, 2]$.

(5) Express $\mathbf{x} = [6, 3, 1]$ as a linear combination of $\mathbf{u} = [1, 1, 1]$, $\mathbf{v} = [1, 1, 0]$, $\mathbf{w} = [1, 0, 0]$. i.e., show that $\mathbf{x} = \mathbf{u} + 2\mathbf{v} + 3\mathbf{w}$

(6) Prove that $\{[3, 1, -4]^t,\ [2, 5, 6]^t,\ [1, 4, 8]^t\}$ is a basis for R^3. **Hint:**

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Let A=[3,1,4; 2,5,6; 1,4,8]; and show that det(A) is not equal to 0.
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(7) Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^3$, and let $L : R^3 \to R^2$ be a linear transformation such that $L(\mathbf{x}) = [1, 0]^t$, $L(\mathbf{y}) = [0, 1]^t$, $L(\mathbf{z}) = [1, -1]^t$. Then $L(2\mathbf{x} - 3\mathbf{y} + 4\mathbf{z}) = 2L(\mathbf{x}) - 3L(\mathbf{y}) + 4L(\mathbf{z}) = [6, -7]^t$.

(8) Let $L : R^3 \to R^2$ be a linear transformation such that $L([1, 0, 0]^t) = [1, 1]^t$, $L([0, 1, 0]^t) = [1, -1]^t$, $L([0, 0, 1]^t) = [1, 0]^t$. Then $Ker(L) = \{\alpha[1, 1, -2]^t\} = span([1, 1, -2]^t)$