H2: Determinants

(1) $A = \begin{bmatrix} 2 & 5 & 1 \\ 1 & 0 & -1 \\ 4 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}$

(a) Show that $det(A) = -14, \quad det(B) = 0, \quad det(C) = 1$

(b) Find the adjoints of matrices A, B, C, respectively.

(c) Use Cramer's rule to find A^{-1} and C^{-1}, respectively.

(2) Let $D = \begin{bmatrix} 2 & 4 & 3 & 1 \\ 0 & 5 & 6 & 3 \\ -1 & 2 & 4 & -2 \\ 7 & 0 & 1 & 3 \end{bmatrix}$, $E = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 5 & 6 \\ 3 & 3 & 7 & 9 \\ 3 & 5 & 6 & 10 \end{bmatrix}$, $F = \begin{bmatrix} 1 & 6 & 11 & 16 & 21 \\ 2 & 7 & 12 & 17 & 22 \\ 3 & 8 & 13 & 18 & 23 \\ 4 & 9 & 14 & 19 & 24 \\ 5 & 10 & 15 & 20 & 25 \end{bmatrix}$

(a) Show that $det(D) = 340, \quad det(E) = 1, \quad det(F) = 0$

(3) Let $A \in R^{n \times n}$, if there exists an $\mathbf{x} \neq \mathbf{0}$ in R^n such that $A\mathbf{x} = \lambda \mathbf{x}$. λ is called *an eigenvalue*. For a small $n \leq 3$, we can directly compute the eigenvalues of a matrix A by solving the n-degree characteristic polynomial equation $det(\lambda I - A) = 0$.

(a) Give the characteristic polynomial equation of matrices A, B, and C given in problem **(1)**.

(b) Solve the polynomial equations in part **(a)**.

(4) Let $P \in R^{n \times n}$. If $P^2 = P$ and $P \neq I$, show that $det(P) = 0$.

(5) Let $H \in R^{n \times n}$. If $H^2 = I$, then $det(H) = 1$ *or* -1. Find an $H \neq \pm I$ such that $det(H) = -1$.

(*Hint*: a Householder matrix, $H = I - 2\mathbf{u}\mathbf{u}^t$, where $\|\mathbf{u}\|_2 = 1$)

(4) Let $P \in R^{n\times n}$. If $P^2 = P$ and $P \neq I$, show that $det(P) = 0$.

(Proof) By contradiction, suppose that $det(P) \neq 0$, then P is invertible, then $P^2 = P \rightarrow P(P-I) = O$ implies that $P - I = O$, which contradicts $P \neq I$.

(5) Let $H \in R^{n\times n}$. If $H^2 = I$, then $det(H) = 1 \; or - 1$. Find an $H \neq \pm I$ such that $det(H) = -1$.

($Hint$: a Householder matrix, $H = I - 2\mathbf{u}\mathbf{u}^t$, where $\|\mathbf{u}\|_2 = 1$)

(Ans) Let $\mathbf{u} = [1/2,\ \sqrt{3}/2]^t$, then $\}\mathbf{u}\|_2 = 1$, and then $H = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$.