# H1 : Linear Systems of Equations

**(1)** Let $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$.

**(a)** Find $AB$ and $BA$.

**(b)** Find $(BA)^t$ and $A^tB^t$.

**(c)** Find $(BA)^{-1}$. Can you find $(AB)^{-1}$?

**(2)** A linear system of equations is given below.

$$\begin{array}{rcrcrcr} 2x & - & 2y & + & 3z & = & 5 \\ -2x & + & 3y & - & 4z & = & -6 \\ 4x & - & 3y & + & 7z & = & 11 \end{array}$$

**(a)** Write this equation as $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

**(b)** What is the augmented matrix for this system?

**(c)** Apply Gaussian elimination and back substitution to solve $A\mathbf{x} = \mathbf{b}$.

**(d)** Find $A = LU$, where $L$ is unit lower$-\Delta$ and $U$is upper$-\Delta$.

**(3)** Let $\mathbf{e}_j \in R^n$ be a unit vector with $j$-th component 1 and 0 otherwise, e.g., $\mathbf{e}_2 = [0, 1, 0]^t$ for $n = 3$. Let the matrices $P, Q, R$ be defined as

$$P = I + 2\mathbf{e}_2\mathbf{e}_1^t, \quad Q = I - 3\mathbf{e}_3\mathbf{e}_1^t, \quad R = I + 4\mathbf{e}_3\mathbf{e}_2^t.$$

**(a)** Compute $P^{-1}$, $Q^{-1}$, and $R^{-1}$.

**(b)** Compute $R^{-1}Q^{-1}P^{-1}$ and $(RQP)^{-1}$, respectively.

**(4)** Given

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 7 \\ 1 & 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 0 & 1 \\ 1 & 3 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & 7 \\ -5 & 3 & 5 \end{bmatrix}.$$

**(a)** Find an elementary matrix $E$ such that $EA = B$.

**(b)** Find an elementary matrix $F$ such that $AF = C$.

**(5)** A linear system of equations is given below.

$$\begin{array}{rcrcrcl} x & + & y & & & = & 1 \\ x & + & y & + & 2z & = & 3 \\ 2x & + & y & - & 2z & = & 0 \end{array}$$

**(a)** Write this equation as $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

**(b)** What is the augmented matrix for this system?

**(c)** Apply Gaussian elimination with Partial Pivoting, followed by back substitution to solve $A\mathbf{x} = \mathbf{b}$.

**(d)** Find a permutation matrix $P$ such that $PA = LU$, where $L$ is unit lower$-\Delta$ and $U$is upper$-\Delta$.

**(6)** Let $C, D \in R^{n\times n}$ be invertible. prove the following equalities (Exercise 243 on P.87).

**(a)** $(C^{-1} + D^{-1})^{-1} = C(C + D)^{-1}D$

**(b)** $(I + CD)^{-1}C = C(I + DC)^{-1}$

**(c)** $(C + DD^t)^{-1}D = C^{-1}D(I + D^tC^{-1}D)^{-1}$

# Solutions for Exercise 1

**(1)** $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$.

$(a)$ $AB = \begin{bmatrix} 3 & 12 & 6 \\ 5 & -2 & 8 \\ 4 & 5 & 7 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 10 \\ 13 & 7 \end{bmatrix}$

$(b)$ $(BA)^t = \begin{bmatrix} 1 & 13 \\ 10 & 7 \end{bmatrix}$, $A^tB^t = (BA)^t)$, $(c)$ $(BA)^{-1} = \frac{1}{123}\begin{bmatrix} -7 & 10 \\ 13 & -1 \end{bmatrix}$, $(AB)^{-1}$ does not exist.

**(2)** A linear system of equations is given below.

$$\begin{array}{rcrcrcr} 2x & - & 2y & + & 3z & = & 1 \\ -2x & + & 3y & - & 4z & = & 0 \\ 4x & - & 3y & + & 7z & = & 5 \end{array}$$

$(a)$ $\begin{bmatrix} 2 & -2 & 3 \\ -2 & 3 & -4 \\ 4 & -3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$, $(b)$ $\left[\begin{array}{ccc|c} 2 & -2 & 3 & 1 \\ -2 & 3 & -4 & 0 \\ 4 & -3 & 7 & 5 \end{array}\right]$

$(d)$ $LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow (c)$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

**(3)** Let $\mathbf{e}_j \in R^n$ be a unit vector with $j$-th component 1 and 0 otherwise, e.g., $\mathbf{e}_2 = [0,1,0]^t$ for $n = 3$. Let the matrices $P, Q, R$ be defined as

$$P = I + 2\mathbf{e}_2\mathbf{e}_1^t, \quad Q = I - 3\mathbf{e}_3\mathbf{e}_1^t, \quad R = I + 4\mathbf{e}_3\mathbf{e}_2^t$$

$$(a)\quad P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad R^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$

$$(b)\quad R^{-1}Q^{-1}P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 11 & -4 & 1 \end{bmatrix}, \quad P^{-1}Q^{-1}R^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}$$

$$(c)\quad P = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

**(4)**

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**5** $\mathbf{x} = [1,0,1]^t$.

# Matlab Solutions

(1)

```
      A=[ 3, 0; -1, 2; 1, 1];
      B=[ 1, 4, 2; 3, 1, 5];
  (a) AB=A*B,   BA=B*A
  (b) BAt=(B*A)'
  (c) BAinv=inv(BA)
```

(3)

```
      e1=[1, 0, 0]';
      e2=[0, 1, 0]';
      e3=[0, 0, 1]';
  (a) Pinv=inv(P), Qinv=inv(Q), Rinv=inv(R)
  (b) Rinv*Qinv*Pinv; Pinv*Qinv*Rinv
  (c) P=eye(3)+2*e2*e1'
      Q=eye(3)-3*e3*e1'
      R=eye(3)+4*e3*e2'
  (d) PQRinv=inv(P*Q*R)
```

(5)

```
      A=[1,1,0; 1,1,2; 2,1,-2];
      b=[1;3;0];
      x=A\b
      so x=[1, 0, 1]'
```