# Problems of Eigenvalues/Eigenvectors

- ♣ Reveiw of Eigenvalues and Eigenvectors
- ♣ Gerschgorin's Disk Theorem
- ♣ Power and Inverse Power Methods
- ♣ Jacobi Transform for Symmetric Matrices
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# Definition and Examples

Let $A \in R^{n\times n}$. If $\exists$ $\mathbf{v} \neq \mathbf{0}$ such that $A\mathbf{v} = \lambda\mathbf{v}$, $\lambda$ is called an eigenvalue of matrix $A$, and $\mathbf{v}$ is called an eigenvector corresponding to (or belonging to) the eigenvalue $\lambda$. Note that $\mathbf{v}$ is an eigenvector implies that $\alpha\mathbf{v}$ is also an eigenvector for all $\alpha \neq 0$. We define the Eigenspace($\lambda$) as the vector space spanned by all of the eigenvectors corresponding to the eigenvalue $\lambda$.

$$A\mathbf{x} = \lambda\mathbf{x} \;\Rightarrow\; (\lambda I - A)\mathbf{x} = \mathbf{0},\; \mathbf{x} \neq \mathbf{0} \;\Rightarrow\; det(\lambda I - A) = P(\lambda) = 0.$$

*Examples:*

1. $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $\lambda_1 = 2$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\lambda_2 = 1$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
2. $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, $\lambda_1 = 2$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\lambda_2 = 1$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
3. $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, $\lambda_1 = 4$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda_2 = 2$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
4. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\lambda_1 = j$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ j \end{bmatrix}$, $\lambda_2 = -j$, $\mathbf{u}_2 = \begin{bmatrix} j \\ 1 \end{bmatrix}$, $j = \sqrt{-1}$.
5. $B = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$, then $\lambda_1 = 3$, $\mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$; $\lambda_2 = -1$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
6. $C = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$, then $\tau_1 = 4$, $\mathbf{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$; $\tau_2 = 2$, $\mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

Note that $\|\mathbf{u}_i\|_2 = 1$ and $\|\mathbf{v}_i\|_2 = 1$ for $i = 1, 2$. Denote $U = [\mathbf{u}_1, \mathbf{u}_2]$ and $V = [\mathbf{v}_1, \mathbf{v}_2]$, then

$$U^{-1}BU = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}, \quad V^{-1}CV = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

Note that $V^t = V^{-1}$ but $U^t \neq U^{-1}$.

$$\Sigma_{j=1}^{n} \lambda_j = \Sigma_{i=1}^{n} a_{ii} \ \ and \ \Pi_{j=1}^{n} \lambda_j = det(A)$$

Let $A \in R^{n\times n}$, then $P(\lambda) = det(\lambda I - A)$ is called the *characteristic polynomial* of matrix $A$.

□ *Fundamental Theorem of Algebra*

A real polynomial $P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$ of degree $n$ has $n$ roots $\{\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n\}$ such that

$$P(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n) = \lambda^n - \left(\sum_{i=1}^{n} \lambda_i\right)\lambda^{n-1} + \cdots + (-1)^n \left(\prod_{i=1}^{n} \lambda_i\right)$$

- $\sum_{i=1}^{n} \lambda_i \ = \ \sum_{i=1}^{n} a_{ii} \ = \ tr(A) \ \ (called the trace of A)$
- $\prod_{i=1}^{n} \lambda_i \ = \ det(A)$

□ *Gershgorin's Disk Theorem*

Every eigenvalue of matrix $A \in R^{n\times n}$ lies in at least one of the following disks

$$D_i = \{x \mid |x - a_{ii}| \ \le \sum_{j\neq i} |a_{ij}|\}, \quad 1 \le i \le n$$

*Example:* $B = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 4 & 1 \\ 2 & 2 & 5 \end{bmatrix}, \quad \lambda_1, \lambda_2, \lambda_3 \in D_1 \cup D_2 \cup D_3$, where

$D_1 = \{z \mid \ |z-3| \le 2\},\ D_2 = \{z \mid \ |z-4| \le 1\},\ D_3 = \{z \mid \ |z-5| \le 4\}.$

Note that $\lambda_1 = 6.5616, \quad \lambda_2 = 3.0000, \quad \lambda_3 = 2.4383.$

□ A matrix is said to be *diagonally dominant* if $|a_{ii}| > \sum_{j\neq i} |a_{ij}|, \ \forall\ 1 \le i \le n.$

◇ A diagonally dominant matrix is invertible.

**Theorem:** Let $A, P \in R^{n\times n}$, with $P$ nonsingular, then $\lambda$ is an eigenvalue of $A$ with eigenvector $\mathbf{x}$ iff $\lambda$ is an eigenvalue of $P^{-1}AP$ with eigenvector $P^{-1}\mathbf{x}$.

**(Proof)** Let $\mathbf{x}$ be an eigenvector of $A$ corresponding to the eigenvalue $\lambda$, that is, $A\mathbf{x} = \lambda\mathbf{x}$. Then, we have

$$(P^{-1}AP)(P^{-1}\mathbf{x}) = P^{-1}A(PP^{-1})\mathbf{x} = P^{-1}A\mathbf{x} = P^{-1}(\lambda\mathbf{x}) = \lambda(P^{-1}\mathbf{x})$$

Thus, $P^{-1}\mathbf{x}$ is an eigenvector corresponding to the eigenvalue $\lambda$ of the matrix $P^{-1}AP$ (according to the definition).

On the other hand,

$$(P^{-1}AP)(P^{-1}\mathbf{x}) = \lambda(P^{-1}\mathbf{x})$$

implies that $A\mathbf{x} = \lambda\mathbf{x}$ could be achieved based on simple matrix operations.

**Theorem:** Let $A \in R^{n\times n}$ and let $\lambda$ be an eigenvalue of $A$ with eigenvector $\mathbf{x}$. Then

**(a)** $\alpha\lambda$ is an eigenvalue of matrix $\alpha A$ with eigenvector $\mathbf{x}$

**(b)** $\lambda - \mu$ is an eigenvalue of matrix $A - \mu I$ with eigenvector $\mathbf{x}$

**(c)** If $A$ is nonsingular, then $\lambda \neq 0$ and $\lambda^{-1}$ is an eigenvalue of $A^{-1}$ with eigenvector $\mathbf{x}$

Let $\mathbf{x}$ be an eigenvector of $A$ corresponding to the eigenvalue $\lambda$, that is, $A\mathbf{x} = \lambda\mathbf{x}$. Then

**Proof of (a)** $(\alpha A)\mathbf{x} = \alpha(A\mathbf{x}) = \alpha(\lambda\mathbf{x}) = (\alpha\lambda)\mathbf{x}$.

**Proof of (b)** $(A - \mu I)\mathbf{x} = A\mathbf{x} - \mu\mathbf{x} = \lambda\mathbf{x} - \mu\mathbf{x} = (\lambda - \mu)\mathbf{x}$.

**Proof of (c)** If $A$ is nonsingular, none of its eigenvalues is zero, otherwise, $A\mathbf{x} = \lambda\mathbf{x} = 0 \cdot \mathbf{x} = \mathbf{0}$ and $\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$ which implies that $\mathbf{x} = \mathbf{0}$ that contradicts that $\mathbf{x}$ is an eigenvector (of $A$). Then, $A\mathbf{x} = \lambda\mathbf{x}$ implies that $\frac{1}{\lambda}\mathbf{x} = A^{-1}\mathbf{x}$. Therefore, $\frac{1}{\lambda}$ is an eigenvalue of matrix $A^{-1}$ with eigenvector $\mathbf{x}$.

**Definition:** A matrix $A$ is similar to $B$, denote by $A \sim B$, iff there exists an invertible matrix $U$ such that $U^{-1}AU = B$. Furthermore, a matrix $A$ is *orthogonally similar* to $B$, iff there exists an orthogonal matrix $Q$ such that $Q^tAQ = B$.

**Theorem:** Two similar matrices have the same eigenvalues, i.e., $A \sim B \Rightarrow \lambda(A) = \lambda(B)$.

**Proof** Since $A \sim B$, we have $B = U^{-1}AU$ for some $U$, then

$$|\lambda I - B| = |U^{-1}(\lambda I)U - U^{-1}AU| = |U^{-1}(\lambda I - A)U| = |U^{-1}|\cdot|\lambda I - A|\cdot|U| = |U|^{-1}\cdot|\lambda I - A|\cdot|U|$$

# Diagonalization of Matrices

**Theorem:** Suppose $A \in R^{n\times n}$ has $n$ linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $V = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$, then $V^{-1}AV = diag[\lambda_1, \lambda_2, \ldots, \lambda_n]$.

◇ If $A \in R^{n\times n}$ has $n$ distinct eigenvalues, then their corresponding eigenvectors are linearly independent. Thus, any matrix with distinct eigenvalues can be diagonalized.

◇ Not all matrices have distinct eigenvalues, therefore not all matrices are diagonalizable.

**Nondiagonalizable Matrices**

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

**Diagonalizable Matrices**

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

**Spectrum Decomposition Theorem:** Every real symmetric matrix can be orthogonally diagonalized.

◇ $U^tAU = \Lambda$ or $A = U\Lambda U^t = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i \mathbf{u}_i^t$, where $U$ is an orthogonal matrix, and $\Lambda = diag[\lambda_1, \lambda_2, \cdots, \lambda_n]$.

# Similarity transformation and triangularization

**Schur's Theorem:** $\forall\, A \in R^{n\times n}$, $\exists$ an orthogonal matrix $U$ such that $U^tAU = T$ is upper-$\Delta$. The eigenvlues must be shared by the similarity matrix $T$ and appear along its main diagonal.

**Hint:** By induction, suppose that the theorem has been proved for all matrices of order $n-1$, and consider $A \in R^{n\times n}$ with $A\mathbf{x} = \lambda\mathbf{x}$ and $\|\mathbf{x}\|_2 = 1$, then $\exists$ a Householder matrix $H_1$ such that $H_1\mathbf{x} = \beta\mathbf{e}_1$, *e.g.*, $\beta = -\|\mathbf{x}\|_2$, hence

$$H_1AH_1^t\mathbf{e}_1 = H_1A(H_1^{-1}\mathbf{e}_1) = H_1A(\beta^{-1}\mathbf{x}) = H_1\beta^{-1}A\mathbf{x} = \beta^{-1}\lambda(H_1\mathbf{x}) = \beta^{-1}\lambda(\beta\mathbf{e}_1) = \lambda\mathbf{e}_1$$

Thus,

$$H_1AH_1^t = \begin{bmatrix} \lambda & | & * \\ --- & | & --- \\ O & | & A^{(1)} \end{bmatrix}$$

**Spectrum Decomposition Theorem:** Every real symmetric matrix can be orthogonally diagonalized.

$\diamond$ $U^tAU = \Lambda$ or $A = U\Lambda U^t = \sum_{i=1}^n \lambda_i \mathbf{u}_i\mathbf{u}_i^t$, where $U$ is an orthogonal matrix, and $\Lambda = diag[\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n]$.

**Definition:** A symmetric matrix $A \in R^{n\times n}$ is nonnegative definite if $\mathbf{x}^tA\mathbf{x} \geq 0\ \forall\ \mathbf{x} \in R^n$, $\mathbf{x} \neq \mathbf{0}$.

**Definition:** A symmetric matrix $A \in R^{n\times n}$ is positive definite if $\mathbf{x}^tA\mathbf{x} > 0\ \forall\ \mathbf{x} \in R^n$, $\mathbf{x} \neq \mathbf{0}$.

**Singular Value Decomposition Theorem:** Each matrix $A \in R^{m\times n}$ can be decomposed as $A = U\Sigma V^t$, where both $U \in R^{m\times m}$ and $V \in R^{n\times n}$ are orthogonal. Moreover, $\Sigma \in R^{m\times n} = diag[\sigma_1, \sigma_2, \ldots, \sigma_k, 0, \ldots, 0]$ is essentially diagonal with the singular values satisfying $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k > 0$.

$\diamond$ $A = U\Sigma V^t = \sum_{i=1}^k \sigma_i \mathbf{u}_i\mathbf{v}_i^t$

*Example:*

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},\quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

# A Jacobi Transform (Givens Rotation)

$$J(i,k;\theta) = \begin{bmatrix} 1 & \cdot & \cdot & \cdots & \cdot & \cdot & 0 \\ 0 & \ddots & \cdot & \cdots & \cdot & \vdots & 0 \\ 0 & \cdot & c & \cdots & s & \cdot & 0 \\ \cdot & \vdots & \cdot & \ddots & \cdot & \vdots & \cdot \\ 0 & \cdot & -s & \cdots & c & \cdot & 0 \\ 0 & \vdots & \cdot & \cdots & \cdot & \ddots & 0 \\ \cdot & \cdot & 0 & \cdots & 0 & \cdot & 1 \end{bmatrix}$$

$J_{hh} = 1$ if $h \neq i$ or $h \neq k$, where $i < k$

$J_{ii} = J_{kk} = c = \cos\theta$

$J_{ki} = -s = -\sin\theta,\ J_{ik} = s = \sin\theta$

Let $\mathbf{x}, \mathbf{y} \in R^n$, then $\mathbf{y} = J(i,k;\theta)\mathbf{x}$ implies that

$y_i = cx_i + sx_k$

$y_k = -sx_i + cx_k$

$c = \frac{x_i}{\sqrt{x_i^2 + x_k^2}},\ s = \frac{x_k}{\sqrt{x_i^2 + x_k^2}},$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \quad then \quad J(2,4;\theta)\mathbf{x} = \begin{bmatrix} 1 \\ \sqrt{20} \\ 3 \\ 0 \end{bmatrix}$$

# Jacobi Transforms (Givens Rotations)

The Jacobi method consists of a sequence of orthogonal similarity transformations such that

$$J_K^t J_{K-1}^t \cdots J_2^t J_1^t A J_1 J_2 \cdots J_{K-1} J_K = \Lambda$$

where each $J_i$ is orthogonal, so is $Q = J_1 J_2 \cdots J_{K-1} J_K$.

Each Jacobi transform (Given rotation) is just a plane rotation designed to annihilate one of the off-diagonal matrix elements. Let $A = (a_{ij})$ be symmetric, then

$B = J^t(p, q, \theta) A J(p, q, \theta)$, where

$$b_{rp} = ca_{rp} - sa_{rq} \quad for \;\; r \neq p, \; r \neq q$$

$$b_{rq} = sa_{rp} + ca_{rq} \quad for \;\; r \neq p, \; r \neq q$$

$$b_{pp} = c^2 a_{pp} + s^2 a_{qq} - 2sca_{pq}$$

$$b_{qq} = s^2 a_{pp} + c^2 a_{qq} + 2sca_{pq}$$

$$b_{pq} = (c^2 - s^2) a_{pq} + sc(a_{pp} - a_{qq})$$

To set $b_{pq} = 0$, we choose $c, s$ such that

$$\alpha = \cot(2\theta) = \frac{c^2 - s^2}{2sc} = \frac{a_{qq} - a_{pp}}{2a_{pq}} \tag{1}$$

For computational convenience, let $t = \frac{s}{c}$, then $t^2 + 2\alpha t - 1 = 0$ whose smaller root (in absolute sense) can be computed by

$$t = \frac{sgn(\alpha)}{\sqrt{\alpha^2 + 1} + |\alpha|}, \quad and \quad c = \frac{1}{\sqrt{1 + t^2}}, \quad s = ct, \quad \tau = \frac{s}{1 + c} \tag{2}$$

Remark

$$b_{pp} = a_{pp} - ta_{pq}$$

$$b_{qq} = a_{qq} + ta_{pq}$$

$$b_{rp} = a_{rp} - s(a_{rq} + \tau a_{rp})$$

$$b_{rq} = a_{rq} + s(a_{rp} - \tau a_{rq})$$

# Algorithm of Jacobi Transforms to Diagonalize A

$A^{(0)} \leftarrow A$

for $k = 0, 1, \cdots,$ until convergence

Let $|a_{pq}^{(k)}| = Max_{i<j}\{|a_{ij}^{(k)}|\}$

Compute

$\alpha_k = \frac{a_{qq}^{(k)} - a_{pp}^{(k)}}{2a_{pq}^{(k)}}$, solve $\cot(2\theta_k) = \alpha_k$ for $\theta_k$.

$t = \frac{sgn(\alpha)}{\sqrt{\alpha^2+1}+|\alpha|}$

$c = \frac{1}{\sqrt{1+t^2}},$ , $s = ct$

$\tau = \frac{s}{1+c}$

$A^{(k+1)} \leftarrow J_k^t A^{(k)} J_k$, where $J_k = J(p, q, \theta_k)$

endfor

# Convergence of Jacobi Algorithm to Diagonalize A

**Proof:**

Since $|a_{pq}^{(k)}| \geq |a_{ij}^{(k)}|$ for $i \neq j,\ p \neq q$, then

$|a_{pq}^{(k)}|^2 \geq off(A^{(k)})/2N$, where $N = \frac{n(n-1)}{2}$, and

$off(A^{(k)}) = \sum_{i \neq j}^{n} \left(a_{ij}^{(k)}\right)^2$, the sum of square off-diagonal elements of $A^{(k)}$

Furthermore,

$$\begin{aligned} off(A^{(k+1)}) &= off(A^{(k)}) - 2\left(a_{pq}^{(k)}\right)^2 + 2\left(a_{pq}^{(k+1)}\right)^2 \\ &= off(A^{(k)}) - 2\left(a_{pq}^{(k)}\right)^2, \quad since \ \ a_{pq}^{(k+1)} = 0 \\ &\leq off(A^{(k)})\left(1 - \tfrac{1}{N}\right), \quad since |a_{pq}^{(k)}|^2 \geq off(A^{(k)}/2N \end{aligned}$$

Thus

$$off(A^{(k+1)}) \leq \left(1 - \frac{1}{N}\right)^{k+1} off(A^{(0)}) \quad \to \ 0 \ \ as \ \ k \ \to \ \infty$$

Example:

$$A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad J(1,2;\theta) = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then

$$A^{(1)} = J^t(1,2;\theta)AJ(1,2;\theta) = \begin{bmatrix} 4c^2 - 4cs + 3s^2 & 2c^2 + cs - 2s^2 & -s \\ 2c^2 + cs - 2s^2 & 3c^2 + 4cs + 4s^2 & c \\ -s & c & 1 \end{bmatrix}$$

Note that $off(A^{(1)}) = 2 < 10 = off(A^{(0)}) = off(A)$

# Example for Convergence of Jacobi Algorithm

$$A^{(0)} = \begin{bmatrix} 1.0000 & 0.5000 & 0.2500 & 0.1250 \\ 0.5000 & 1.0000 & 0.5000 & 0.2500 \\ 0.2500 & 0.5000 & 1.0000 & 0.5000 \\ 0.1250 & 0.2500 & 0.5000 & 1.0000 \end{bmatrix}, \quad A^{(1)} = \begin{bmatrix} 1.5000 & 0.0000 & 0.5303 & 0.2652 \\ 0.0000 & 0.5000 & 0.1768 & 0.0884 \\ 0.5303 & 0.1768 & 1.0000 & 0.5000 \\ 0.2652 & 0.0884 & 0.5000 & 1.0000 \end{bmatrix}$$

$$A^{(2)} = \begin{bmatrix} 1.8363 & 0.0947 & 0.0000 & 0.4917 \\ 0.0947 & 0.5000 & 0.1493 & 0.0884 \\ 0.0000 & 0.1493 & 0.6637 & 0.2803 \\ 0.4917 & 0.0884 & 0.2803 & 1.0000 \end{bmatrix}, \quad A^{(3)} = \begin{bmatrix} 2.0636 & 0.1230 & 0.1176 & 0.0000 \\ 0.1230 & 0.5000 & 0.1493 & 0.0405 \\ 0.1176 & 0.1493 & 0.6637 & 0.2544 \\ 0.0000 & 0.0405 & 0.2544 & 0.7727 \end{bmatrix}$$

$$A^{(4)} = \begin{bmatrix} 2.0636 & 0.1230 & 0.0915 & 0.0739 \\ 0.1230 & 0.5000 & 0.0906 & 0.1254 \\ 0.0915 & 0.0906 & 0.4580 & 0.0000 \\ 0.0739 & 0.1254 & 0.0000 & 0.9783 \end{bmatrix}, \quad A^{(5)} = \begin{bmatrix} 2.0636 & 0.1018 & 0.0915 & 0.1012 \\ 0.1018 & 0.4691 & 0.0880 & 0.0000 \\ 0.0915 & 0.0880 & 0.4580 & 0.0217 \\ 0.1012 & 0.0000 & 0.0217 & 1.0092 \end{bmatrix}$$

$$A^{(6)} = \begin{bmatrix} 2.0701 & 0.0000 & 0.0969 & 0.1010 \\ 0.0000 & 0.4627 & 0.0820 & -0.0064 \\ 0.0969 & 0.0820 & 0.4580 & 0.0217 \\ 0.1010 & -0.0064 & 0.0217 & 1.0092 \end{bmatrix}, \quad A^{(15)} = \begin{bmatrix} 2.0856 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.5394 & 0.0000 & -0.0000 \\ 0.0000 & 0.0000 & 0.3750 & 0.0000 \\ 0.0000 & -0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

# Cholesky Algorithm

□ **Theorem:** Every positive definitive matrix $A$ can be decomposed as $A = LL^t$, where $L$ is $lower - \Delta$.

□ **Algorithm:** $A \in R^{n \times n}$, $A = LL^t$, $A$ is positive definite and $L$ is $lower - \Delta$.

for $j = 0, 1, \cdots, n-1$

$L_{jj} \leftarrow \left[A_{jj} - \sum_{k=0}^{j-1} L_{jk}^2\right]^{1/2}$

for $i = j+1, j+2, \cdots, n-1$

$L_{ij} \leftarrow \left[A_{ij} - \sum_{k=0}^{j-1} L_{ik} L_{jk}\right] / L_{jj}$

endfor

endfor

$$C = \begin{bmatrix} 4 & -2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = L_1 L_1^t$$

$$A = \begin{bmatrix} 9 & 3 & -3 \\ 3 & 17 & 3 \\ -3 & 3 & 27 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{bmatrix} = L_2 L_2^t$$

# Power of A Matrix and Its Eigenvalues

**Theorem:** Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be eigenvalues of $A \in R^{n\times n}$. Then $\lambda_1^k, \lambda_2^k, \cdots, \lambda_n^k$ are eigenvalues of $A^k \in R^{n\times n}$ with the same corresponding eigenvectors of $A$. That is,

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i \quad \rightarrow \quad A^k \mathbf{v}_i = \lambda_i^k \mathbf{v}_i \;\; \forall\, 1 \leq i \leq n$$

Suppose that the matrix $A \in R^{n\times n}$ has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_1, \cdots, \mathbf{v}_n$ corresponding to eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then any $\mathbf{x} \in R^n$ can be written as

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

Then

$$A^k\mathbf{x} = \lambda_1^k c_1\mathbf{v}_1 + \lambda_2^k c_2\mathbf{v}_2 + \cdots + \lambda_n^k c_n\mathbf{v}_n$$

In particular, if $|\lambda_1| > |\lambda_j|$ for $2 \leq j \leq n$ and $c_1 \neq 0$, then $A^k\mathbf{x}$ will tend to lie in the direction $\mathbf{v}_1$ when $k$ is *large enough.*

# Power Method for Computing the Largest Eigenvalues

Suppose that the matrix $A \in R^{n\times n}$ is diagonalizable and that $U^{-1}AU = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$ with $U = [\mathbf{v}_1,\ \mathbf{v}_2,\ \cdots,\ \mathbf{v}_n]$ and $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$. Given $\mathbf{u}^{(0)} \in R^n$, then power method produces a sequence of vectors $\mathbf{u}^{(k)}$ as follows.

for $k = 1, 2, \cdots$

$\mathbf{z}^{(k)} = A\mathbf{u}^{(k-1)}$

$r^{(k)} = z_m^{(k)} = \|\mathbf{z}^{(k)}\|_\infty$, for some $1 \leq m \leq n$.

$\mathbf{u}^{(k)} = \mathbf{z}^{(k)}/r^{(k)}$

endfor

$\lambda_1$ *must be real since the complex eigenvalues must appear in a "relatively conjugate pair".*

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{c} \lambda_1 = 3 \\ \\ \lambda_2 = 1 \end{array}, \quad \mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Let $\mathbf{u}^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\mathbf{u}^{(5)} = \begin{bmatrix} 1.0 \\ 0.9918 \end{bmatrix}$, and $r^{(5)} = 2.9756$.

# QR Iterations for Computing Eigenvalues

```
%
% Script File: shiftQR.m
% Solving Eigenvalues by shift-QR factorization
%
Nrun=15;
fin=fopen('dataMatrix.txt');
fgetL(fin); % read off the header line
n=fscanf(fin,'%d',1);
A=fscanf(fin,'%f',[n n]);
A=A';
SaveA=A;
for k=1:Nrun,
  s=A(n,n);
  A=A-s*eye(n);
  [Q R]=qr(A);
  A=R*Q+s*eye(n);
end
eig(SaveA)
%
%  dataMatrix.txt
%
Matrices for computing eigenvalues by QR factorization or shift-QR
  5
  1.0    0.5   0.25  0.125  0.0625
  0.5    1.0   0.5   0.25   0.125
  0.25   0.5   1.0   0.5    0.25
  0.125  0.25  0.5   1.0    0.5
  0.0625 0.125 0.25  0.5    1.0
  4               for shift-QR studies
  2.9766  0.3945  0.4198  1.1159
  0.3945  2.7328 -0.3097  0.1129
  0.4198 -0.3097  2.5675  0.6079
  1.1159  0.1129  0.6097  1.7231
```

# Norms of Vectors and Matrices

**Definition:** A vector norm on $R^n$ is a function

$$\tau \;:\; R^n \;\rightarrow\; R^+ = \{x \geq 0 |\ x \in R\}$$

that satisfies

**(1)** $\tau(\mathbf{x}) > 0 \;\; \forall\ \mathbf{x} \neq \mathbf{0},\ \tau(\mathbf{0}) = 0$

**(2)** $\tau(c\mathbf{x}) = |c|\tau(\mathbf{x})\ \forall\ c \in R,\ \mathbf{x} \in R^n$

**(3)** $\tau(\mathbf{x}+\mathbf{y}) \leq \tau(\mathbf{x}) + \tau(\mathbf{y})\ \forall\ \mathbf{x}, \mathbf{y} \in R^n$

*Hölder norm (p-norm)* $\|\mathbf{x}\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}\ \ for\ p \geq 1.$

**(p=1)** $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$ (Mahattan or City-block distance)

**(p=2)** $\|\mathbf{x}\|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2}$ (Euclidean distance)

**(p=$\infty$)** $\|\mathbf{x}\|_\infty = max_{1\leq i \leq n}\{|x_i|\}$ ($\infty$-norm)

**Definition:** A matrix norm on $R^{m\times n}$ is a function

$$\tau \ : \ R^{m\times n} \ \rightarrow \ R^{+} = \{x \geq 0 | \ x \in R\}$$

that satisfies

**(1)** $\tau(A) > 0 \ \ \forall \ A \neq O, \tau(O) = 0$

**(2)** $\tau(cA) = |c|\tau(A) \ \forall \ c \in R, \ \ A \in R^{m\times n}$

**(3)** $\tau(A+B) \leq \tau(A) + \tau(B) \ \ \forall \ A, B \in R^{m\times n}$

*Consistency Property:* $\tau(AB) \leq \tau(A)\tau(B) \ \ \forall \ A, B$

**(a)** $\tau(A) = max\{|a_{ij}| \mid 1 \leq i \leq m, \ 1 \leq j \leq n\}$

**(b)** $\|A\|_F = \left[\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2\right]^{1/2}$ (Fröbenius norm)

**Subordinate Matrix Norm:** $\|A\| = max_{\|\mathbf{x}\| \neq \mathbf{0}}\{\|A\mathbf{x}\| / \|\mathbf{x}\|\}$

**(1)** If $A \in R^{m\times n}$, then $\|A\|_1 = max_{1\leq j\leq n} \left(\sum_{i=1}^{m} |a_{ij}|\right)$

**(2)** If $A \in R^{m\times n}$, then $\|A\|_\infty = max_{1\leq i\leq m} \left(\sum_{j=1}^{n} |a_{ij}|\right)$

**(3)** Let $A \in R^{n\times n}$ be real symmetric, then $\|A\|_2 = max_{1\leq i\leq n}|\lambda_i|$, where $\lambda_i \in \lambda(A)$

**Theorem:** Let $\mathbf{x} \in R^n$ and let $A = (a_{ij}) \in R^{n\times n}$. Define $\|A\|_1 = Sup_{\|\mathbf{u}\|_1=1}\{\|A\mathbf{u}\|_1\}$

**Proof:** For $\|\mathbf{u}\|_1 = 1$,

$$\|A\|_1 = Sup\{\|A\mathbf{u}\|_1\} = \sum_{i=1}^{n} |\sum_{j=1}^{n} a_{ij}u_j| \leq \sum_{j=1}^{n}\sum_{i=1}^{n} |a_{ij}||u_j| = \sum_{j=1}^{n} |u_j| \sum_{i=1}^{n} |a_{ij}|$$

Then

$$\|A\|_1 \leq Max_{1\leq j\leq n}\{\sum_{i=1}^{n} |a_{ij}|\} \sum_{j=1}^{n} |u_j| = Max_{1\leq j\leq n}\{\sum_{i=1}^{n} |a_{ij}|\}$$

On the other hand, let $\sum_{i=1}^{n} |a_{ik}| = Max_{1\leq j\leq n}\{\sum_{i=1}^{n} |a_{ij}|\}$ and choose $\mathbf{u} = \mathbf{e}_k$, which completes the proof.

**Theorem:** Let $A = [a_{ij}] \in R^{m\times n}$, and define $\|A\|_\infty = Max_{\|\mathbf{u}\|_\infty=1}\{\|A\mathbf{u}\|_\infty\}$.

Show that $\|A\|_\infty = Max_{1\leq i\leq m}\left\{\sum_{j=1}^{n}|a_{ij}|\right\}$

**Proof:** Let $\sum_{j=1}^{n}|a_{Kj}| = Max_{1\leq i\leq m}\left\{\sum_{j=1}^{n}|a_{ij}|\right\}$, for any $\mathbf{x} \in R^n$ with $\|\mathbf{x}\|_\infty = 1$, we have

$$\begin{aligned}
\|A\mathbf{x}\|_\infty &= Max_{1\leq i\leq m}\left\{|\textstyle\sum_{j=1}^{n} a_{ij}x_j|\right\} \\
&\leq Max_{1\leq i\leq m}\left\{\textstyle\sum_{j=1}^{n} |a_{ij}|\cdot|x_j|\right\} \leq Max_{1\leq i\leq m}\left\{\textstyle\sum_{j=1}^{n} |a_{ij}|\|\mathbf{x}\|_\infty\right\} \\
&\leq Max_{1\leq i\leq m}\left\{\textstyle\sum_{j=1}^{n} |a_{ij}|\right\} = \textstyle\sum_{j=1}^{n} |a_{Kj}|
\end{aligned}$$

In particular, if we pick up $\mathbf{y} \in R^n$ such that $y_j = sign(a_{Kj})$, $\forall\ 1 \leq j \leq n$, then $\|\mathbf{y}\|_\infty = 1$, and $\|A\mathbf{y}\|_\infty = \sum_{j=1}^{n} |a_{Kj}|$, which completes the proof.

**Theorem:** Let $A = [a_{ij}] \in R^{n\times n}$, and define $\|A\|_2 = Max_{\|\mathbf{x}\|_2=1}\{\|A\mathbf{x}\|_2\}$. Show that

$$\|A\|_2 = \sqrt{\rho(A^tA)} = \sqrt{maximum\ eigenvalue\ of\ A^tA}\ \ (spectral\ radius)$$

(**Proof**) Let $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n$ be eigenvalues and their corresponding unit eigenvectors $\mathbf{u}_1,\ \mathbf{u}_2,\ \cdots,\ \mathbf{u}_n$ of matrix $A^tA$, that is,

$$(A^tA)\mathbf{u}_i = \lambda_i\mathbf{u}_i\ \ and\ \ \|\mathbf{u}_i\|_2 = 1\ \ \forall\ 1 \le i \le n.$$

Since $\mathbf{u}_1,\ \mathbf{u}_2,\ \cdots,\ \mathbf{u}_n$ must be an orthonormal basis based on *spectrum decomposition theorem*, for any $\mathbf{x} \in R^n$, we have $\mathbf{x} = \sum_{i=1}^{n} c_i\mathbf{u}_i$. Then

$$\begin{aligned}
\|A\|_2 &= Max_{\|\mathbf{x}\|_2=1}\{\|A\mathbf{x}\|_2\} \\
&= \sqrt{Max_{\|\mathbf{x}\|_2=1}\{\|A\mathbf{x}\|_2^2\}} \\
&= \sqrt{Max_{\|\mathbf{x}\|_2=1}\{\mathbf{x}^tA^tA\mathbf{x}\}} \\
&= \sqrt{Max_{\|\mathbf{x}\|_2=1}|\sum_{i=1}^{n}\lambda_i c_i^2|} \\
&= \sqrt{Max_{1\le j\le n}\{|\lambda_j|\}}
\end{aligned}$$

# A Markov Process

Suppose that 10% of the people outside Taiwan move in, and 20% of the people indside Taiwan move out in each year. Let $y_k$ and $z_k$ be the population at the end of the $k-th$ year, outside Taiwan and inside Taiwan, respectively. Then we have

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} y_{k-1} \\ z_{k-1} \end{bmatrix} \quad \Rightarrow \quad \lambda_1 = 1.0, \ \lambda_2 = 0.7$$

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (0.7)^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

□ A *Markov* matrix A is nonnegative with each colume adding to 1.

- **(a)** $\lambda_1 = 1$ is an eigenvalue with a nonnegative eigenvector $\mathbf{x}_1$.
- **(b)** The other eigenvalues satisfy $|\lambda_i| \leq 1$.
- **(c)** If any power of $A$ has all positive entries, and the other $|\lambda_i| < 1$. Then $A^k \mathbf{u}_0$ approaches the steady state of $\mathbf{u}_\infty$ which is a multiple of $\mathbf{x}_1$ as long as the projection of $\mathbf{u}_0$ in $\mathbf{x}_1$ is not zero.

◇ Check Perron-Fröbenius theorem in Strang's book.

# $e^A$ and Differential Equations

♣ $e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \cdots + \frac{A^m}{m!} + \cdots$

♣ $\frac{du}{dt} = -\lambda u \;\Rightarrow\; u(t) = e^{-\lambda t}u(0)$

♣ $\frac{d\mathbf{u}}{dt} = -A\mathbf{u} \;=\; \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{u} \;\Rightarrow\; \mathbf{u}(t) = e^{-tA}\mathbf{u}(0)$

♣ $A = U\Lambda U^t$ for an orthogonal matrix $U$, then

$$e^A = Ue^{\Lambda}U^{=}Udiag[e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}]U^t$$

♣ Solve $x''' - 3x'' + 2x' = 0$.

Let $y = x'$, $z = y' = x''$, and let $\mathbf{u} = [x, y, z]^t$. The problem is reduced to solving

$$\mathbf{u}' = A\mathbf{u} \;=\; \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix}\mathbf{u}$$

Then

$$\mathbf{u}(t) = e^{tA}\mathbf{u}(0) \;=\; \begin{bmatrix} \frac{1}{\sqrt{21}} & \frac{1}{\sqrt{3}} & 1 \\ \frac{2}{\sqrt{21}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{4}{\sqrt{21}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -2.2913 & 2.2913 \\ 0 & 3.4641 & -1.7321 \\ 1 & -1.5000 & 0.5000 \end{bmatrix}\mathbf{u}(0)$$

# Problems Solved by Matlab

Let A, B, H, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, $\mathbf{b}$ be matrices and vectors defined below, and $H = I - 2\mathbf{u}\mathbf{u}^t$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 6 & 0 \\ -2 & 7 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ 2 \\ -5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

**1.** Let A=LU=QR, find L, U; Q, R.

**2.** Find determinants and inverses of matrices A, B, and H.

**3.** Solve $A\mathbf{x} = \mathbf{b}$, how to find the number of floating-point operations are required?

**4.** Find the ranks of matrices A, B, and H.

**5.** Find the characteristic polynomials of matrices A and B.

**6.** Find 1-norm, 2-norm, and $\infty$-norm of matrices A, B, and H.

**7.** Find the eigenvalues/eigenvectors of matrices A and B.

**8.** Find matrices U and V such that $U^{-1}AU$ and $V^{-1}BV$ are diagonal matrices.

**9.** Find the singular values and singular vectors of matrices A and B.

**10.** Randomly generate a 4×4 matrix C with $0 \le C(i,j) \le 9$.