

Vector Space and Linear Transform

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Definition of A Vector Space

Definition: A vector space V (over R) is a set on which the operations of addition \oplus and scalar multiplication \odot are defined. The set V associated with the operations of addition and scalar multiplication is said to form a *vector space* if the following axioms are satisfied.

$$\text{(A1)} \quad \mathbf{x} \oplus \mathbf{y} = \mathbf{y} \oplus \mathbf{x} \quad \forall \mathbf{x}, \mathbf{y} \in V$$

$$\text{(A2)} \quad (\mathbf{x} \oplus \mathbf{y}) \oplus \mathbf{z} = \mathbf{x} \oplus (\mathbf{y} \oplus \mathbf{z}) \quad \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V$$

$$\text{(A3)} \quad \exists \mathbf{0} \in V \text{ such that } \mathbf{x} \oplus \mathbf{0} = \mathbf{x} \quad \forall \mathbf{x} \in V$$

$$\text{(A4)} \quad \forall \mathbf{x} \in V, \exists -\mathbf{x} \in V \text{ such that } \mathbf{x} \oplus (-\mathbf{x}) = \mathbf{0}$$

$$\text{(A5)} \quad \alpha \odot (\mathbf{x} \oplus \mathbf{y}) = (\alpha \odot \mathbf{x}) \oplus (\alpha \odot \mathbf{y}), \quad \forall \alpha \in R \text{ and } \mathbf{x}, \mathbf{y} \in V$$

$$\text{(A6)} \quad (\alpha + \beta) \odot \mathbf{x} = (\alpha \odot \mathbf{x}) \oplus (\beta \odot \mathbf{x}), \quad \forall \alpha, \beta \in R \text{ and } \mathbf{x} \in V$$

$$\text{(A7)} \quad (\alpha \cdot \beta) \odot \mathbf{x} = \alpha \odot (\beta \odot \mathbf{x}), \quad \forall \alpha, \beta \in R \text{ and } \mathbf{x} \in V$$

$$\text{(A8)} \quad 1 \odot \mathbf{x} = \mathbf{x} \text{ for a } 1 \in R \text{ and } \forall \mathbf{x} \in V$$

Examples

$$\text{(1)} \quad R^n \text{ (over } R), \text{ in particular, } n = 2, 3$$

$$\text{(2)} \quad C[a, b], \text{ for example, } C[0, 1]$$

$$\text{(3)} \quad P_n = \{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \mid a_j \in R\}$$

$$\text{(4)} \quad R^{m \times n} = \text{the set of all } m \text{ by } n \text{ real matrices}$$

Exercises for Vector Space

1. Mark \bigcirc if the vector addition and scalar multiplication forms a vector space, otherwise mark \times .

(\times) (a) For (R^3, \oplus, \odot) , the set of all triples of real numbers $[x, y, z]$ with the operations

$$[u, v, w] \oplus [x, y, z] = [u + x, v + y, w + z] \quad \text{and} \quad \alpha \odot [x, y, z] = [\alpha x, y, z]$$

(\times) (b) For (R^3, \oplus, \odot) , the set of all triples of real numbers $[x, y, z]$ with the operations

$$[u, v, w] \oplus [x, y, z] = [u + x, v + y, w + z] \quad \text{and} \quad \alpha \odot [x, y, z] = [0, 0, 0]$$

(\times) (c) For (R^2, \oplus, \odot) , the set of all paris of real numbers $[x, y]$ with the operations

$$[u, v] \oplus [x, y] = [u + x, v + y] \quad \text{and} \quad \alpha \odot [x, y] = [2\alpha x, 2\alpha y]$$

(\times) (d) For (R^2, \oplus, \odot) , the set of all paris of real numbers $[x, y]$ with the operations

$$[u, v] \oplus [x, y] = [u + x + 1, v + y + 1] \quad \text{and} \quad \alpha \odot [x, y] = [\alpha x, \alpha y]$$

(\bigcirc) (e) For (V, \oplus, \odot) , where $V = \{[1, y] | y \in R\}$, the set of all paris of real numbers $[1, y]$ with the operations

$$[1, x] \oplus [1, y] = [1, x + y] \quad \text{and} \quad \alpha \odot [1, y] = [1, \alpha y]$$

(\bigcirc) (f) For (V, \oplus, \odot) , where $V = \{x \in R | x > 0\}$, $\alpha \in R$,

$$x \oplus y = xy \quad \text{and} \quad \alpha \odot x = x^\alpha$$

(\bigcirc) (g) For (V, \oplus, \odot) , where $V = \{a + bx | a, b \in R\}$,

$$(a + bx) \oplus (c + dx) = (a + c) + (b + d)x \quad \text{and} \quad \alpha \odot (c + dx) = (\alpha c) + (\alpha d)x$$

Subspaces of Vector Space

Definition: A subspace U of a vector space V is a *nonempty* subset satisfying

$$\mathbf{x} \oplus \mathbf{y} \in U \text{ and } \alpha \odot \mathbf{x} \in U \quad \forall \mathbf{x}, \mathbf{y} \in U; \alpha \in R$$

Examples

The set of lower- Δ (upper- Δ) matrices

The set of tridiagonal (diagonal, Hessenberg) matrices

Let $A \in R^{m \times n}$, $A = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n]$, and $A^t = [\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_m]$, then

$$\text{Null}(A) = \{\mathbf{x} \in R^n \mid A\mathbf{x} = \mathbf{0}\} \subset R^n \text{ (Nullspace)}$$

$$R(A) = \{\sum_{j=1}^n \alpha_j \mathbf{a}_j \mid \alpha_j \in R\} \subset R^m \text{ (Column space)}$$

$$R(A^t) = \{\sum_{i=1}^m \beta_i \mathbf{b}_i \mid \beta_i \in R\} \subset R^n \text{ (Row space)}$$

Theorem: The system $A\mathbf{x} = \mathbf{b}$ is solvable iff the vector \mathbf{b} can be expressed as a linear combination of the columns of A

$$A = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n]$$

$$A\mathbf{x} = \mathbf{b} \text{ iff } \sum_{i=1}^n x_i \mathbf{a}_i = \mathbf{b}$$

Overdetermined, Underdetermined, Homogeneous Systems

$$\begin{array}{cccccc}
 a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
 a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
 & & \cdot & & \cdot & & \cdot & & \cdot \\
 & & \cdot & & \cdot & & \cdot & & \cdot \\
 a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
 \end{array}$$

Definition: A linear system is said to be *overdetermined* if there are more equations than unknowns ($m > n$), *underdetermined* if $m < n$, *homogeneous* if $b_i = 0, \forall 1 \leq i \leq m$.

$$\begin{array}{lll}
 x + y = 1 & x + y = 3 & x + y = 2 \\
 (A) \quad x - y = 3 & (B) \quad x - y = 1 & (C) \quad 2x + 2y = 4 \\
 -x + 2y = -2 & 2x + y = 5 & -x - y = -2
 \end{array}$$

(A) has no solution, (B) has unique solution, (C) has infinitely many solutions

$$\begin{array}{ll}
 (D) \quad x + 2y + z = -1 & x + 2y + z = 5 \\
 2x + 4y + 2z = 3 & (E) \quad 2x - y + z = 3
 \end{array}$$

(D) has no solution, (E) has infinitely many solutions

Solutions of m Equations in n Unknowns

Theorem: $\forall A \in R^{m \times n}$, there corresponds a permutation matrix P , a unit lower- Δ matrix L , and an $m \times n$ upper trapezoidal matrix U such that $PA = LU$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 6 \\ 2 & 5 & 6 & 8 \\ 1 & 3 & 4 & 5 \end{bmatrix} \Rightarrow PA = P_{34}P_{23}A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 1 & 3 & 4 & 5 \\ 1 & 2 & 3 & 6 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 8 \\ 1 & 3 & 4 & 5 \\ 1 & 2 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = LU$$

Linear Span

Definition: Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be vectors in a vector space V . A sum of the form $\sum_{i=1}^n c_i \mathbf{v}_i$, where c_i 's are scalars, is called a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$. The *linear span* is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ and is denoted by

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$$

□ In R^3 , $\text{span}(\mathbf{e}_1, \mathbf{e}_2) = \{[a, b, 0]^t \mid a, b \in R\}$

□ The nullspace could be $\text{span}([1, -2, 1, 0]^t, [-1, 1, 0, 1]^t)$, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

□ $\text{Null}(A) = \text{span}([1, -2, 1, 0]^t, [-1, 1, 0, 1]^t)$

□ $\text{Null}(A) = \text{span}([1, -2, 1, 0]^t, [0, -1, 1, 1]^t)$

Theorem: If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are elements of a vector space V , $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ is a subspace of V .

Proof: Show that $a\mathbf{u} + b\mathbf{v} \in V, \forall a, b \in R; \mathbf{u}, \mathbf{v} \in V$

Spanning Sets

Definition: The set of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a *spanning set* for V iff each $\mathbf{v} \in V$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

- (1) $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, [1, 2, 3]^t\}$ is a spanning set of R^3 .
- (2) $\{[1, 1, 1]^t, [1, 1, 0]^t, [1, 0, 0]^t\}$ is a spanning set of R^3 .
- (3) $\{[1, 0, 1]^t, [0, 1, 0]^t\}$ is not a spanning set of R^3 .
- (4) $\{[1, 2, 4]^t, [2, 1, 3]^t, [4, -1, 1]^t\}$ is not a spanning set of R^3 .
- (5) $\text{span}(1, x, x^2) = \text{span}(1 - x^2, x + 2, x^2)$, where $P_2 = \{ax^2 + bx + c \mid a, b, c \in R\}$

Linear Independence

Definition: The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are said to be *linearly independent* if $\sum_{i=1}^n c_i \mathbf{v}_i = \mathbf{0}$ implies that $c_i = 0$ for $1 \leq i \leq n$. Otherwise, they are said to be linearly dependent.

$$\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha = \beta = 0$$

Let

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{bmatrix}$$

- (1) The column vectors of A are *linearly independent*.
- (2) The column vectors of B are *linearly dependent*.
- (3) The column vectors of C are *linearly dependent*.

Theorem: A set of n vectors in R^m must be linearly dependent if $n > m$

Basis and Dimension

Definition: A *basis* for a vector space is a set of vectors satisfying two properties: (1) it is linearly independent, (2) it spans the vector space.

- $\{\mathbf{e}_1, \mathbf{e}_2\}$ is not a basis for R^3 since $\text{span}(\mathbf{e}_1, \mathbf{e}_2) \neq R^3$
- The vectors $[1, 0]^t, [0, 1]^t, [2, 1]^t$ spans R^2 but are not linearly independent so it is not a basis for R^2

Definition: Any two bases for a vector space V contain the same number of vectors. This number, shared by all bases and expresses the number of freedom of the space, is called the *dimension* of V .

Theorem: Suppose that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ and $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ are both bases for the same vector space S , then $m = n$.

Theorem: Any linearly independent set in a vector space V can be extended to a basis by adding more vectors if necessary. Any spanning set in V can be reduced to a basis by discarding vectors if necessary.

Example: Let $A \in R^{64 \times 17}$ be a matrix of rank 11.

(1) $6 = (17 - 11)$ independent vectors \mathbf{x} satisfy $A\mathbf{x} = \mathbf{0}$

(2) $53 = (64 - 11)$ independent vectors \mathbf{y} satisfy $A^t\mathbf{y} = \mathbf{0}$

The Rank of A Matrix

□ The rank of a matrix $A \in R^{m \times n}$ can be defined as the number of linear independent columns. In Matlab command:

`rank(A)`

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -1 & 1 & 1 \\ 1 & 1 & 2 & -1 \end{bmatrix}$$

Suppose the Gaussian elimination reduces $A\mathbf{x}=\mathbf{b}$ to $U\mathbf{x}=\mathbf{c}$ with r pivots, i.e., the last $m - r$ rows are zero. Then, there is a solution only if the last $m - r$ components of \mathbf{c} are also zero. If $m = r$, there is always a solution. The general solution is the sum of a particular solution (with all free variables zero) and a homogeneous solution (with $n - r$ free variables as independent parameters). If $r = n$, there are no free variables and the nullspace contains only $\mathbf{x}=\mathbf{0}$. The number r is called the rank of matrix A .

Suppose \mathbf{x}_p satisfies $A\mathbf{x}_p = \mathbf{b}$ and \mathbf{x}_h satisfies $A\mathbf{x}_h = \mathbf{0}$

Then $\mathbf{x}_g = \mathbf{x}_p + \mathbf{x}_h$ satisfies $A\mathbf{x}_g = A\mathbf{x}_p + A\mathbf{x}_h = \mathbf{b} + \mathbf{0} = \mathbf{b}$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

Four Fundamental Subspaces from a Matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow E = \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

matrix \Rightarrow row echelon form \Rightarrow reduced row echelon form

♣ **Fundamental Theorem of Linear Algebra:** Let $A \in R^{m \times n}$ have rank r ,

- (1) $R(A)$: the column space of A , $\dim(R(A)) = r$
- (2) $N(A)$: the nullspace of A , $\dim(N(A)) = n - r$
- (3) $R(A^t)$: the row space of A (the column space of A^t), $\dim(R(A^t)) = r$
- (4) $N(A^t)$: the left nullspace of A (the column space of A^t), $\dim(N(A^t)) = m - r$

- $N(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$
- $R(A) = \{\sum_{j=1}^n t_j \mathbf{a}_j \mid A = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n]\}$

The row space of A has the same dimension r as the row space of U because $R(A^t) = R(U^t)$. The nullspace $N(A)$ has dimension $n - r$.

- (1) $\dim(R(A)) + \dim(N(A)) = r + (n - r) = n$
- (2) $\dim(R(A^t)) + \dim(N(A^t)) = r + (m - r) = m$

Example: $A \in R^{3 \times 4}$

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

$$\begin{aligned} \dim(R(A)) &= 2 \\ \dim(N(A)) &= 4 - 2 \\ \dim(R(A^t)) &= 2 \\ \dim(N(A^t)) &= 3 - 2 \end{aligned}$$

Vector Norms

Definition: A vector norm on R^n is a function

$$\tau : R^n \rightarrow R^+ = \{x \geq 0 \mid x \in R\}$$

that satisfies

(1) $\tau(\mathbf{x}) > 0 \quad \forall \mathbf{x} \neq \mathbf{0}, \quad \tau(\mathbf{0}) = 0$

(2) $\tau(c\mathbf{x}) = |c|\tau(\mathbf{x}) \quad \forall c \in R, \quad \mathbf{x} \in R^n$

(3) $\tau(\mathbf{x} + \mathbf{y}) \leq \tau(\mathbf{x}) + \tau(\mathbf{y}) \quad \forall \mathbf{x}, \mathbf{y} \in R^n$

Hölder norm (p-norm) $\|\mathbf{x}\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$.

(p=1) $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$ (Mahattan or City-block distance)

(p=2) $\|\mathbf{x}\|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2}$ (Euclidean distance)

(p=∞) $\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} \{|x_i|\}$ (∞-norm)

Matrix Norms

Definition: A matrix norm on $R^{m \times n}$ is a function

$$\tau : R^{m \times n} \rightarrow R^+ = \{x \geq 0 \mid x \in R\}$$

that satisfies

- (1) $\tau(A) > 0 \quad \forall A \neq O, \tau(O) = 0$
- (2) $\tau(cA) = |c|\tau(A) \quad \forall c \in R, A \in R^{m \times n}$
- (3) $\tau(A + B) \leq \tau(A) + \tau(B) \quad \forall A, B \in R^{m \times n}$

Consistency Property: $\tau(AB) \leq \tau(A)\tau(B) \quad \forall A, B$

- (a) $\tau(A) = \max\{|a_{ij}| \mid 1 \leq i \leq m, 1 \leq j \leq n\}$
- (b) $\|A\|_F = \left[\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2 \right]^{1/2}$ (Fröbenius norm)

Subordinate Matrix Norm: $\|A\| = \max_{\|\mathbf{x}\| \neq 0} \{\|A\mathbf{x}\| / \|\mathbf{x}\|\}$

- (1) If $A \in R^{m \times n}$, then $\|A\|_1 = \max_{1 \leq j \leq n} (\sum_{i=1}^m |a_{ij}|)$
- (2) If $A \in R^{m \times n}$, then $\|A\|_\infty = \max_{1 \leq i \leq m} (\sum_{j=1}^n |a_{ij}|)$
- (3) Let $A \in R^{n \times n}$ be real symmetric, then $\|A\|_2 = \max_{1 \leq i \leq n} |\lambda_i|$, where $\lambda_i \in \lambda(A)$

Linear Transformation

Definition: A mapping L from a vector space V to a vector space W is said to be a linear transform (transformation) or a linear operator if

$$L(\alpha \cdot \mathbf{v}_1 + \beta \cdot \mathbf{v}_2) = (\alpha \odot L(\mathbf{v}_1)) \oplus (\beta \odot L(\mathbf{v}_2)), \quad \forall \alpha, \beta \in R, \mathbf{v}_1, \mathbf{v}_2 \in V$$

Examples: Projection, Scaling, Rotation, Reflection on $V = R^2$

(a) $L(\mathbf{x}) = \mathbf{u}^t \mathbf{x}$, for $\mathbf{u} \in V$

(b) $L(\mathbf{x}) = s\mathbf{x}$, for $s \in R$

(c) $L(\mathbf{x}) = R_\theta \mathbf{x}$, where $R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

(d) $L(\mathbf{x}) = \mathbf{y}$, where $y_1 = -x_1$ and $y_2 = x_2$

(e) $L(f) = F$, where $f \in C[a, b]$ and $F(x) = \int_a^x f(t) dt$

(f) $L(f) = f'$, where $f \in C^1[a, b]$ and $f'(x) = \frac{d}{dx} f(x)$

(g) $L(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x} \in R^n$, $A \in R^{m \times n}$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Image and Kernel

Let $L : V \rightarrow W$ be a linear transform, and let $S \subset V$ be a *subspace* of V . The *kernel* of L , denoted by $\text{Ker}(L)$, is defined by

$$\text{Ker}(L) = \{\mathbf{v} \in V \mid L(\mathbf{v}) = \mathbf{0}\}$$

The *image* of S under L , denoted by $L(S)$, is defined by

$$L(S) = \{\mathbf{w} \in W \mid \mathbf{w} = L(\mathbf{v}) \text{ for some } \mathbf{v} \in S\}$$

Theorem: Let $L : V \rightarrow W$ be a linear transform, and let $S \subset V$ be a *subspace* of V , then

- (a) $\text{Ker}(L)$ is a subspace of V
- (b) $L(S)$ is a subspace of W

Changing Coordinates in R^2

$$\{\mathbf{e}_1, \mathbf{e}_2\} \Rightarrow \{\mathbf{v}_1, \mathbf{v}_2\}$$

Any vector in $\mathbf{w} \in R^2$ can be expressed as $\mathbf{w} = x\mathbf{e}_1 + y\mathbf{e}_2 = [x, y]^t$, suppose that we want to express \mathbf{w} as a linear combination of \mathbf{v}_1 and \mathbf{v}_2 as $\mathbf{w} = x'\mathbf{v}_1 + y'\mathbf{v}_2 = [x', y']^t$. What are $\{x, y\}$ and $\{x', y'\}$ related?

$$\begin{bmatrix} x \\ y \end{bmatrix} = x\mathbf{e}_1 + y\mathbf{e}_2 = x'\mathbf{v}_1 + y'\mathbf{v}_2 = [\mathbf{v}_1, \mathbf{v}_2] \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = [\mathbf{v}_1, \mathbf{v}_2]^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example: $\mathbf{v}_1 = [1, 1]^t$, $\mathbf{v}_2 = [-1, 1]^t$, then

$$[\mathbf{v}_1, \mathbf{v}_2] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow [\mathbf{v}_1, \mathbf{v}_2]^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$$

Gauss Transform

Define an elementary matrix as

$$E_{ik}(r) = I - r\mathbf{e}_i\mathbf{e}_k^t, \quad i > k \quad \Rightarrow \quad E_{ik}(r)^{-1} = I + r\mathbf{e}_i\mathbf{e}_k^t$$

A *Gauss* transform is a matrix of the form

$$\prod_{i=n}^{k+1} E_{ik} = E_{nk}E_{n-1,k} \cdots E_{k+1,k}$$

which can annihilate the components of a vector \mathbf{x} after index k .

Examples

$$G = E_{31}(-1)E_{21}(2) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \quad \Rightarrow \quad G\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Householder Transform (Elementary Reflector)

$$H = I - 2\mathbf{u}\mathbf{u}^t, \text{ where } \mathbf{u} \in R^n \text{ with } \|\mathbf{u}\|_2 = 1$$

$$H^t = H \text{ and } H^{-1} = H$$

Let $\mathbf{x} = [3, 1, 5, 1]^t$, then $\|\mathbf{x}\|_2 = \sqrt{3^2 + 1^2 + 5^2 + 1^2} = 6$.

Define $\mathbf{v} = \mathbf{x} + \|\mathbf{x}\|_2\mathbf{e}_1$, and let $\mathbf{u} = \mathbf{v}/\|\mathbf{v}\|_2$, then

$$H = I - 2\mathbf{u}\mathbf{u}^t = \frac{1}{54} \begin{bmatrix} -27 & -9 & -45 & -9 \\ -9 & 53 & -5 & -1 \\ -45 & -5 & 29 & -5 \\ -9 & -1 & -5 & 53 \end{bmatrix}, \text{ and } H\mathbf{x} = \begin{bmatrix} -6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Jacobi Transform (Givens' Rotation)

$$J(i, k; \theta) = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & c & \cdot & -s & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & s & \cdot & c & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & 0 & \cdot & 0 & \cdot & 1 \end{bmatrix}$$

$$J_{hh} = 1 \text{ if } h \neq i \text{ or } h \neq k, \text{ where } i < k$$

$$J_{ii} = J_{kk} = c = \cos \theta$$

$$J_{ki} = s = \sin \theta, J_{ik} = -s = -\sin \theta$$

Let $\mathbf{x}, \mathbf{y} \in R^n$, then $\mathbf{y} = J(i, k; \theta)\mathbf{x}$ implies that

$$y_i = cx_i - sx_k$$

$$y_k = sx_i + cx_k$$

$$c = \frac{x_i}{\sqrt{x_i^2 + x_k^2}}, \quad s = \frac{-x_k}{\sqrt{x_i^2 + x_k^2}},$$

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}, \quad \text{then } J(2, 4; \theta)\mathbf{x} = \begin{bmatrix} 1 \\ \sqrt{20} \\ 3 \\ 0 \end{bmatrix}$$

Affine Transform with Applications

$$\mathbf{y} = A\mathbf{x} + \mathbf{t} \quad \Rightarrow \quad \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} e_i \\ f_i \end{bmatrix}$$

w	a	b	c	d	e	f	$ ad - bc $
1	0	0	0	0.16	0	0	0.01
2	0.85	0.04	-0.04	0.85	0	1.60	0.85
3	0.20	-0.26	0.23	0.22	0	1.60	0.07
4	-0.15	0.28	0.26	0.24	0	0.44	0.07

Table 1: An IFS consisting of 4 affine transforms for Fern

Textures Generated by Fractal Models

Fractal models used to generate such textures as *ferns*, *Sierpinski triangles*, and *snowflakes* have recently received attention in many image compression field. Synthesis is based on the iterated function system (IFS) codes [1,2,3], which are nothing but a set of affine transformations. Let $A \in R^{2 \times 2}$ and $\mathbf{t} \in R^2$. An affine transform on $\mathbf{x} \in R^2$ is defined as $A\mathbf{x} + \mathbf{t}$. To describe the algorithm, we denote $A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$, with $p_i = |a_i d_i - b_i c_i| \neq 0$, and let \mathbf{t}_i denote $\begin{bmatrix} e_i \\ f_i \end{bmatrix}$. An algorithm based on IFS codes with K affine transforms is listed below. Experiments conducted using two sets of affine transformations to generate textures are given. The parameters of this fractal model are given in Tables 1 and 2, respectively. Two such synthesized textures are shown in the following Figure.

Table 1. IFS codes for a fern

i	a_i	b_i	c_i	d_i	e_i	f_i
1	0	0	0	0.16	0	0
2	0.85	0.04	-0.04	0.85	0	1.60
3	0.20	-0.26	0.23	0.22	0	1.60
4	-0.15	0.28	0.26	0.24	0	0.44

Table 2. IFS codes for Sierpinski triangles

i	a_i	b_i	c_i	d_i	e_i	f_i
1	0.5	0	0	0.5	0	0
2	0.5	0	0	0.5	1.0	0
3	0.5	0	0	0.5	0.5	0.5

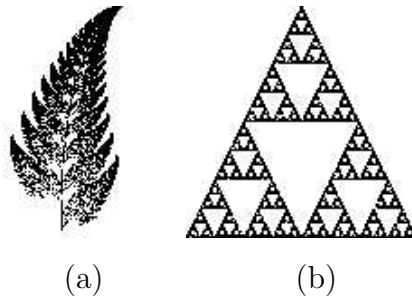


Figure 1: Textures synthesized using IFS codes.

Fractal Generating Algorithm

- (1) Set $m=0$ and randomly pick up an initial point $\mathbf{x}^{(0)} \in R^2$.
- (2) Select $\{A_j, \mathbf{t}_j\}$ according to the probability distribution of $\{r_j, 1 \leq j \leq K\}$, where $r_j = p_j / \sum_{i=1}^K p_i$ for $1 \leq j \leq K$.
- (3) $\mathbf{x}^{(m+1)} \leftarrow A_j \mathbf{x}^{(m)} + \mathbf{t}_j$.
- (4) $m \leftarrow m + 1$.
- (5) Repeat steps 2, 3, 4 until "convergence," for example, $m = 1000$, is achieved.
- (6) Plot $\{\mathbf{x}^{(i)}\}$ for $i = L$ to 1000, say $L=100$.

Convergence of this algorithm was studied by Barnsley [1], and Chu and Chen [3]. An IFS code consisting of two to five contractive affine transforms has been suggested [2,3] which generates self-similar images.

References

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