Solutions for Exam 3 for CS2334(01)
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(30%)1. Let

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 4 \\ 3 & 5 \end{bmatrix}$$

(a) Compute eigenvalues and eigenvectors and give a *spectrum decomposition* for matrix A.

$$A = VDV^t, \quad V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Find the *Cholesky factorization* for A.

$$A = LL^t, \quad L = \begin{bmatrix} \sqrt{2} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{\sqrt{6}}{2} \end{bmatrix}$$

(c) Find the *singular value decomposition* for A.

`The SVD for A is the same as the spectrum decomposition`

(d) Find the *LUdecomposition* for matrix C.

$$C = LU = \begin{bmatrix} 1 & 0 \\ \frac{3}{4} & 1 \end{bmatrix} D = \begin{bmatrix} 4 & 4 \\ 0 & 2 \end{bmatrix}$$

(e) Find the *QR factorization* for C.

$$C = QR = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{bmatrix} \begin{bmatrix} 5 & \frac{31}{5} \\ 0 & -\frac{8}{5} \end{bmatrix}$$

(f) Find the *singular value decomposition* for C.

$$C = USV^t = U \begin{bmatrix} \sqrt{33+\sqrt{1025}} & 0 \\ 0 & \sqrt{33-\sqrt{1025}} \end{bmatrix} V^t$$

(30%)2. Let $B \in R^{3\times 3}$ be a matrix defined as follows.

$$B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

(a) Find the *characteristic polynomial* of B.

(Sa) $p(\lambda) = \lambda^3 - 2\lambda^2 - 5\lambda + 6$

(b) Find the eigenvalues $\lambda_1,\ \lambda_2,\ \lambda_3$ and their corresponding *unit* eigenvectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$, that is, $\|\mathbf{v}_i\|_2 = 1,\ 1 \le i \le 3$.

(Sb) $\lambda_1 = -2,\ \lambda_2 = 3,\ \lambda_3 = 1$

$$\mathbf{v}_1 = [1, 0, 0]^t, \quad \mathbf{v}_2 = [0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}]^t, \quad \mathbf{v}_3 = [0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^t$$

(c) Find the eigenvalues of B^{-1} and their corresponding *unit eigenvectors.*

(Sc) The eigenvalues of B^{-1} and corresponding eigenvectors are $\lambda(B^{-1}) = \{\lambda_1^{-1} = -\frac{1}{2}, \lambda_2^{-1} = \frac{1}{3}, \lambda_3^{-1} = 1\}$ and

$$\mathbf{v}_1 = [1, 0, 0]^t, \quad \mathbf{v}_2 = [0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}]^t, \quad \mathbf{v}_3 = [0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]^t$$

(d) Find an othogonal matrix U such that U^tBU is a diagonal matrix.

(Sd) Let $U = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$, then $U^tBU = diag(-2, 3, 1)$.

(e) Find the eigenvalues $\mu_1,\ \mu_2,\ \mu_3$ of matrix $B + 2I$ and their corresponding *unit eigenvectors* $\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3$.

(Se) $\mu_1 = 0,\ \mu_2 = 5,\ \mu_3 = 3$ and their corresponding *unit eigenvectors* are $\mathbf{w}_1 = \mathbf{v}_1,\ \mathbf{w}_2 = \mathbf{v}_2,\ \mathbf{w}_3 = \mathbf{v}_3$.

(f) Find the eigenvalues $\tau_1,\ \tau_2,\ \tau_3$ of matrix $(B - I)^5$.

(Sf) $\tau_1 = -243,\ \tau_2 = 32,\ \tau_3 = 0$.

3.(20%) Mark ○ if the statement is *true*, and mark × otherwise, or give your comments.

(×) **(a)** Cholesky decomposition is suitable for every real and symmetric matrix.

(×) **(b)** Every nonsingular matrix has an LU decomposition.

(○) **(c)** Every nonsingular matrix has a QR factorization.

(×) **(d)** Every real symmetric matrix must have nonnegative eigenvalues.

(×) **(e)** Every real diagonally dominant matrix must have positive eigenvalues.

(○) **(f)** Let $\mathbf{x}, \mathbf{y} \in R^n$ and $C \in R^{n\times n}$ is an orthogonal matrix, then $\langle C\mathbf{x}, C\mathbf{x}\rangle = \|\mathbf{x}\|_2^2$.

(○) **(g)** The product of all eigenvalues of a real matrix equals its determinant.

(×) **(h)** The product of two orthogonal and symmetric matrices is orthogonal and symmetric.

(○) **(i)** In a QR-factorization $A = QR$ of a nonsingular matrix A, $det(R) \neq 0$.

(×) **(j)** Let $H_i \in R^{n\times n}$, $1 \le i \le k$ be Householder matrices, then $\prod_{j=1}^{k} H_j = (-1)^n$.

(×) **(k)** Let $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ be an orthonormal basis of R^n. If $C \in R^{n\times n}$ is nonsingular, then $\{C\mathbf{v}_1, C\mathbf{v}_2, \cdots, C\mathbf{v}_n\}$ is also an orthonormal basis of R^n.

4.(10%) Let $H \in R^{n\times n}$ be a Householder matrix.

(a) Show that H must have an eigenvalue -1.

(b) Show that if λ is an eigenvalue of H, then $|\lambda| = 1$.

(Sa) : Since $H = I - 2\mathbf{u}\mathbf{u}^t$ for some $\mathbf{u} \in R^{n\times n}$ with $\|\mathbf{u}\|_2 = 1$, we have $H\mathbf{u} = -\mathbf{u}$, therefore, H must have an eigenvalue -1 associated an eigenvector $\mathbf{u}$.

(Sb) Let $\mathbf{x}$ be an eigenvector corresponding to the eigenvalue λ of H so we have $H\mathbf{x} = \lambda\mathbf{x}$. Because H is a Householder matrix, then H is orthogonal. We have

$$\langle H\mathbf{x}, H\mathbf{x}\rangle = (H\mathbf{x})^t(H\mathbf{x}) = \mathbf{x}^t H^t H\mathbf{x} = \mathbf{x}^t\mathbf{x} = \|\mathbf{x}\|_2^2$$

On the other hand,

$$\langle H\mathbf{x}, H\mathbf{x}\rangle = \langle \lambda\mathbf{x}, \lambda\mathbf{x}\rangle = \lambda^2(\mathbf{x}^t\mathbf{x}) = \lambda^2\|\mathbf{x}\|_2^2$$

Therefore, $\lambda^2 = 1$ and $|\lambda| = 1$.

6.(14%) Let $b = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. *Writing* **a single Matlab command** *to solve each of the following questions* for **a** $\sim$ **h** and answer the questions for **i** $\sim$ **j**.

(a) Randomly generate a 3 by 3 matrix A whose elements are integers in [0, 10].

```
A=randi([0,10],3,3); or A=fix(10*random('unif',0,1,3,3))
```

(b) Input vector b.

```
b = [1; -5; 4]
```

(c) Solve the linear system $Ax = b$ for x.

```
x = A\b
```

(d) Input matrix C given above.

```
C = [-2,1,0; 1,-2,0; 0,0,2]
```

(e) Find the characteristic polynomial for C.

```
p=poly(C)
```

(f) Find the eigenvalues and eigenvectors of C.

```
[U, D]=eig(C)
```

(g) Find the $QR - factorization$ of the matrix C.

```
[Q, R]=qr(C)
```

(h) Find the *singular value decomposition* for the matrix C.

```
[U, S, V]=svd(C)
```

(3%)(i) Show the output results of **(f)**.

```
D=diag(-1,-3,2), U holds corresponding eigenvectors.
```

(3%)(j) Show the output results of **(h)**.

```
S=diag(3,2,1), U, V hold left and right singular vectors, respectively.
```