Solutions for Exam 2 for CS2334(01)

December 4, 2017

Name : ____________ *StudentNumber* : ____________ *Index* : ______

I.(20%) Mark ◯ if the statement is *true*, and mark × otherwise.

(×) **(1)** R^m is a vector subspace of R^n for all $1 \leq m \leq n$.

(◯) **(2)** It is possible to find a pair of 2-dimensional subspaes S and T of R^3 such that $dim(S \cap T) = 1$.

(◯) **(3)** If S and T are subspaces of a vector space V, then both $S \cap T$ and $S + T$ are vector subspaces of V.

(×) **(4)** If $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m$ are linearly independent vectors in R^n, then $span(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m) = R^n$.

(◯) **(5)** $rank(A) = rank(A^t)$ for any $A \in R^{m \times n}$.

(◯) **(6)** Let $A \in R^{m \times n}$, then $Null(A) \leq R^n$ and $R(A) \leq R^m$.

(×) **(7)** Let $S = \{[x, y]^t \mid y = 3x + 2\} \subset R^2$, then S is a vector subspace of R^2.

(×) **(8)** If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^n$ and $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{z} \rangle = 0$, then $\mathbf{x} \perp \mathbf{z}$.

(◯) **(9)** A set of n nonzero orthogonal vectors in R^n must be a basis for R^n.

(×) **(10)** Let $U = span([1, 0, 1]^t)$ and $V = span([0, 1, 0]^t)$, then $U \oplus V = R^3$.

II.(40%) *Answer the following questions*

(A) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z} \in R^n$ be orthonormal vectors, then $\|\mathbf{u} - 3\mathbf{v} + 5\mathbf{w} - \mathbf{z}\|_2 =$ __*6*__

(B) Let $\mathbf{x} = [1, 2, 1, 2]^t$, $\mathbf{y} = [1, -1, -1, 1]^t$, then the angle between $\mathbf{x}$ and $\mathbf{y} =$ __$\frac{\pi}{2}$__

(C) Let $V = \{[0, 0, z]^t |\ z \in R\} \subset R^3$, then $V^\perp =$ __$span([1,0,0]^t,\ [0,1,0]^t)$__

(D) Let $\mathbf{u} = [1, 2, 3, 4]^t$, then the rank of $\mathbf{u}\mathbf{u}^t =$ __*1*__

(E) Let $\mathbf{a} = [1, 1, 1]^t$, $\mathbf{b} = [5, 3, 7]^t$, then the projection of $\mathbf{b}$ along the line $\mathbf{a} =$ __5**a**__

(F) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, the least squares error solution of $A\mathbf{x} = \mathbf{b}$ is __$[1, 1]^t$__

(G) The point on the line $y = 2x + 1$ that is closest to $[5, 2]^t$ is __$[\frac{7}{5},\ \frac{19}{5}]^t$__

(H) Let $\mathbf{x} = [1, 1, 1, 1, 1, 1, 1, 1]^t$, $\mathbf{y} = [1, 2, 3, 4, 5, 6, 7, 8]^t$, and $Q \in R^{8\times 8}$ is orthogonal, then $\langle Q\mathbf{x}, Q\mathbf{y}\rangle =$ __*36*__

(I) Let $\mathbf{y} = [-1, 3, -5, 1]^t$, then $\|\mathbf{y}\|_1 + \|\mathbf{y}\|_2 + \|\mathbf{y}\|_\infty =$ __*21*__

(J) Let $f, g \in C[-1, 1]$, and define the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$, then $\langle \sin 2\pi x, \sin 2\pi x\rangle =$ __*1*__, $\langle \cos 4\pi x, \cos 3\pi x\rangle =$ __*0*__

III.(10%) Let $\mathbf{x} \in R^n$, show that $\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1$.

Proof Let $|x_k| = max_{1\le j\le n}\{|x_j|\} = \|\mathbf{x}\|_\infty$, we have

$$|x_k|^2 \le \sum_{j=1}^{n} |x_j|^2 \le \sum_{j=1}^{n} |x_j|^2 + 2 \sum_{1\le i<j\le n} |x_i| \cdot |j_j|$$

which implies that

$$\|\mathbf{x}\|_\infty^2 \le \|\mathbf{x}\|_2^2 \le \|\mathbf{x}\|_1^2$$

Thus,

$$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1$$

which completes the proof.

IV.(10%) Let $L : R^3 \to R^2$ be a linear transformation such that

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(a) Find the kernel of L, $Ker(L)$.

(b) Evaluate $L\left(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right)$.

(Sa)

$$Let\ L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

we have $a = -2c\ and\ b = c$, then $Ker(L) = span([-2, 1, 1]^t)$.

(Sb)

$$L\left(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right) = 3L(\mathbf{e}_1) + 2L(\mathbf{e}_2) + L(\mathbf{e}_3) = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

V.(20%) A Householder matrix H can be defined as $H = I - 2\mathbf{u}\mathbf{u}^t$, where $\mathbf{u} \in R^n$ and $\|\mathbf{u}\|_2 = 1$.

(a) Show that H is symmetric.

(b) Show that H is orthogonal.

(c) Show that $H^{-1} = H$.

(d) Let $\mathbf{u} = [\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}]^t$ and $U = I - 2\mathbf{u}\mathbf{u}^t$, compute U.

(e) Given any two unit vectors $\mathbf{x}, \mathbf{y} \in R^n$, that is, $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = \mathbf{1}$. Find a Householder matrix $T \in R^{n\times n}$ such that $T\mathbf{x} = \mathbf{y}$.

(Sa)

$$H^t = (I - 2\mathbf{u}\mathbf{u}^t)^t = I^t - 2(\mathbf{u}^t)^t\mathbf{u}^t = I - 2\mathbf{u}\mathbf{u}^t = H$$

Thus, H is symmetric.

(Sb)

$$\begin{aligned} H^tH &= (I - 2\mathbf{u}\mathbf{u}^t)^t(I - 2\mathbf{u}\mathbf{u}^t) &&= (I - 2\mathbf{u}\mathbf{u}^t)(I - 2\mathbf{u}\mathbf{u}^t) \\ &= I - 4\mathbf{u}\mathbf{u}^t + 4(\mathbf{u}\mathbf{u}^t)(\mathbf{u}\mathbf{u}^t) &&= I - 4\mathbf{u}\mathbf{u}^t + 4\mathbf{u}(\mathbf{u}^t\mathbf{u})\mathbf{u}^t \\ &= I - 4\mathbf{u}\mathbf{u}^t + 4\mathbf{u}\mathbf{u}^t &&= I \end{aligned}$$

Thus, H is orthogonal.

(Sc) Since $H^tH = HH = I$ by (**a**) and (**b**), thus, $H^{-1} = H$.

(Sd)

$$U = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

(Se) Let $\mathbf{v} = \mathbf{x} - \mathbf{y}$, define $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|_2}$, and $H = I - 2\mathbf{u}\mathbf{u}^t$, then $H\mathbf{x} = \mathbf{y}$.