Solutions for Exam 1 of CS2334 (01)

October 23, 2017

(15%)(1) Given

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 2 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & -1 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}$$

(a) Find AB **(b)** Find $(AB)^{-1}$ **(c)** Find B^tA^t

$$(a)\quad AB = \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} \qquad (b)\quad (AB)^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{5}{4} \end{bmatrix} \qquad (c)\quad B^tA^t = \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}.$$

(15%)(2) Let $C, D \in R^{3\times 3}$ with $det(C) = 4$ and $det(D) = 5$. Find the value of

(a) $det(CD)$ **(b)** $det(3C)$ **(c)** $det(2CD)$ **(d)** $det(C^{-1}D)$

(a) $det(CD) = 20$ **(b)** $det(3C) = 108$ **(c)** $det(2CD) = 160$ **(d)** $det(C^{-1}D) = \frac{5}{4}$

(15%)(3) Let the matrices W and V be defined by the following Matlab commands, respectively.

```
W=[1, 3, 9; 1, 4, 16; 1, 5, 25];
V=[1, 12, 144; 1,13,169; 1, 14, 196];
```

(a) Give the matrices W and V, respectively.

(b) Find $det(W)$

(c) Find $det(V)$

$$(a)\quad W = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix}; \quad (b)\quad det(W) = 2; \quad (c)\quad det(V) = 2.$$

(15%)(4) Let $P, Q, R \in R^{3\times 3}$ be defined as

$$P = I - \mathbf{e}_2\mathbf{e}_1^t, \quad Q = I - 2\mathbf{e}_3\mathbf{e}_1^t, \quad R = I - 3\mathbf{e}_3\mathbf{e}_2^t$$

(a)

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad R^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

(b,c)

$$S = PQR = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix}, \quad T = R^{-1}Q^{-1}P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & 3 & 1 \end{bmatrix}$$

5(a) The augmented matrix is $[A \mid \mathbf{b}]$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

5(b,c) $E_{23}(1)E_{13}(1)E_{12}(-2)[A|\mathbf{b}] = [U|\mathbf{c}]$, $L = (E_{12}(-2))^{-1}(E_{13}(1))^{-1}(E_{23}(1))^{-1}$, where

$$E_{12}(-2) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{13}(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad E_{23}(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \quad [U|\mathbf{c}] = \begin{bmatrix} 2 & 1 & 1 & | & 5 \\ 0 & -8 & -2 & | & -12 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$$

Then $[x, y, z] = [1, 1, 2]$ with $z = 2,\ y = 1,\ x = 1$ are found sequentially (backward).

5(d)

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.5 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & -6 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

5(f) `A=[2,1,1; 4,-6,0; -2,7,2]; b=[5, -2, 9]'; x=A\b`

(15%)(6) Prove the following statements.

(a) A matrix A is *idempotent* if $A^2 = A$. Show that if A is *idempotent*, so is $I - A$.

(b) Show that if A is *idempotent*, then $2A - I$ is invertible and is its own inverse.

(c) Let U_n be the $n \times n$ matrix, each of whose entries is 1. Show that for $n > 1$,

$$(I - U_n)^{-1} = I - \frac{1}{n-1} U_n$$

(a) $(I - A)^2 = I^2 - A - A + A^2 = I - A - A + A = I - A$ which completes the proof.

(b) $(2A - I)(2A - I) = 4A^2 - 2A - 2A + I^2 = 4A - 4A + I = I$ which completes the proof.

(c) Since $U_n^2 = nU_n$, we have

$$\begin{aligned} (I - U_n)(I - \tfrac{1}{n-1} U_n) &= I - U_n - \tfrac{1}{n-1} U_n + \tfrac{1}{n-1} U_n^2 \\ &= I - \tfrac{n}{n-1} U_n + \tfrac{n}{n-1} U_n \\ &= I \end{aligned}$$

Therefore,

$$(I - U_n)^{-1} = I - \frac{1}{n-1} U_n$$