Exam 2 for CS2334(01)

December 4, 2017

Name : ____________ *StudentNumber* : ______________ *Index* : ______

I.(20%) Mark ◯ if the statement is *true*, and mark $\times$ otherwise.

(   ) **(1)** R^m is a vector subspace of R^n for all $1 \leq m \leq n$.

(   ) **(2)** It is possible to find a pair of 2-dimensional subspaes S and T of R^3 such that $dim(S \cap T) = 1$.

(   ) **(3)** If S and T are subspaces of a vector space V, then both $S \cap T$ and $S + T$ are vector subspaces of V.

(   ) **(4)** If $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m$ are linearly independent vectors in R^n, then $span(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m) = R^n$.

(   ) **(5)** $rank(A) = rank(A^t)$ for any $A \in R^{m\times n}$.

(   ) **(6)** Let $A \in R^{m\times n}$, then $Null(A) \leq R^n$ and $R(A) \leq R^m$.

(   ) **(7)** Let $S = \{[x, y]^t |\ y = 3x + 2\} \subset R^2$, then S is a vector subspace of R^2.

(   ) **(8)** If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^n$ and $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{z} \rangle = 0$, then $\mathbf{x} \perp \mathbf{z}$.

(   ) **(9)** A set of n nonzero orthogonal vectors in R^n must be a basis for R^n.

(   ) **(10)** Let $U = span([1, 0, 1]^t)$ and $V = span([0, 1, 0]^t)$, then $U \oplus V = R^3$.

II.(40%) *Answer the following questions*

(A) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z} \in R^n$ be orthonormal vectors, then $\|\mathbf{u} - 3\mathbf{v} + 5\mathbf{w} - \mathbf{z}\|_2 =$ ________

(B) Let $\mathbf{x} = [1, 2, 1, 2]^t$, $\mathbf{y} = [1, -1, -1, 1]^t$, then the angle between $\mathbf{x}$ and $\mathbf{y} =$ ________

(C) Let $V = \{[0, 0, z]^t | \ z \in R\} \subset R^3$, then $V^{\perp} =$ ________

(D) Let $\mathbf{u} = [1, 2, 3, 4]^t$, then the rank of $\mathbf{u}\mathbf{u}^t =$ ________

(E) Let $\mathbf{a} = [1, 1, 1]^t$, $\mathbf{b} = [5, 3, 7]^t$, then the projection of $\mathbf{b}$ along the line $\mathbf{a} =$ ________

(F) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, the least squares error solution of $A\mathbf{x} = \mathbf{b}$ is ________

(G) The point on the line $y = 2x + 1$ that is closest to $[5, 2]^t$ is ________

(H) Let $\mathbf{x} = [1, 1, 1, 1, 1, 1, 1, 1]^t$, $\mathbf{y} = [1, 2, 3, 4, 5, 6, 7, 8]^t$, and $Q \in R^{8\times 8}$ is orthogonal, then $\langle Q\mathbf{x}, Q\mathbf{y}\rangle =$ ________

(I) Let $\mathbf{y} = [-1, 3, -5, 1]^t$, then $\|\mathbf{y}\|_1 + \|\mathbf{y}\|_2 + \|\mathbf{y}\|_\infty =$ ________

(J) Let $f, g \in C[-1, 1]$, and define the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$, then $\langle \sin 2\pi x, \sin 2\pi x\rangle =$ ________, $\langle \cos 4\pi x, \cos 3\pi x\rangle =$ ________

III.(10%) Let $\mathbf{x} \in R^n$, show that $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$.

IV.(10%) Let $L : R^3 \to R^2$ be a linear transformation such that

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(a) Find the kernel of L, $Ker(L)$.

(b) Evaluate $L\left(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right)$.

V.(20%) A Householder matrix H can be defined as $H = I - 2\mathbf{u}\mathbf{u}^t$, where $\mathbf{u} \in R^n$ and $\|\mathbf{u}\|_2 = 1$.

(a) Show that H is symmetric.

(b) Show that H is orthogonal.

(c) Show that $H^{-1} = H$.

(d) Let $\mathbf{u} = [\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}]^t$ and $U = I - 2\mathbf{u}\mathbf{u}^t$, compute U.

(e) Given any two unit vectors $\mathbf{x}, \mathbf{y} \in R^n$, that is, $\|\mathbf{x}\|_\mathbf{2} = \|\mathbf{y}\|_\mathbf{2} = \mathbf{1}$. Find a Householder matrix $T \in R^{n \times n}$ such that $T\mathbf{x} = \mathbf{y}$.