I. (20%) Mark ⃝ if the statement is true, and mark × otherwise.

(a) $R^m$ is a vector subspace of $R^n$ for all $1 \leq m \leq n$.

(b) For $\forall \mathbf{x} = [x_1, x_2]^t, \mathbf{y} = [y_1, y_2]^t$ in $V = R^2$ and $\alpha \in R$, the vector addition $\oplus$ and scalar multiplication $\odot$ are defined as $\mathbf{x} \oplus \mathbf{y} = [x_1 + y_1, x_2 + y_2]^t$ and $\alpha \odot \mathbf{x} = [\alpha x_1, x_2]^t$, respectively. Then $V$ is a vector space.

(c) For $\forall \mathbf{x} = [x_1, x_2]^t, \mathbf{y} = [y_1, y_2]^t$ in $V = R^2$ and $\alpha \in R$, the vector addition $\oplus$ and scalar multiplication $\odot$ are defined as $\mathbf{x} \oplus \mathbf{y} = [x_1 + y_1, x_2 + y_2]^t$ and $\alpha \odot \mathbf{x} = [2\alpha x_1, 2\alpha x_2]^t$, respectively. Then $V$ is a vector space.

(d) Let $S = \{[x, y]^t| y = 3x - 2\} \subset R^2$, then $S$ is a vector subspace of $R^2$.

(e) Let $A \in R^{m \times n}$, then $\text{Null}(A) \leq R^m$ and $R(A) \leq R^n$.

(f) If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^n$ and $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{z} \rangle = 0$, then $\mathbf{x} \perp \mathbf{z}$.

(g) If $R^n = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n)$, then $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a basis for $R^n$.

(h) A set of $n$ nonzero orthogonal vectors in $R^n$ must be a basis.

(i) Let $U = \text{span}([1, 1]^t)$ and $V = \text{span}([-1, 1]^t)$, then $U \oplus V = R^2$.

(j) Let $H_1, H_2, \cdots, H_k \in R^{n \times n}$ be Householder matrices, then $\prod_{j=1}^{k} H_j$ is symmetric and orthogonal.
II. (30%) Answer each of the following questions.

(A) Prove or disapprove that \{[1, 1, 2]^t, [1, -1, 0]^t, [2, 1, 3]^t\} is a basis for \(\mathbb{R}^3\).

(B) Let

\[
A = \begin{bmatrix}
1 & 2 & -1 & 1 \\
2 & 4 & -3 & 3 \\
1 & 2 & 1 & 5
\end{bmatrix}
\]

What are the rank of \(A\) and the dimension of nullspace of \(A\), respectively?

(C) Let \(x = [4, -3, -2, 1]^t\), what are \(\|x\|_1\), \(\|x\|_2\), \(\|x\|_\infty\), respectively?
(D) Given \( Z = \{[0,0,\gamma]^t|\gamma \in R\} \) and let \( f : R^3 \to R^3 \) be represented as \( f(x) = Ax \), where
\[
A = \begin{bmatrix}
-2 & 1 & 0 \\
1 & -2 & 0 \\
0 & 0 & -2 \\
\end{bmatrix}
\]
What are the kernel of \( g(x) = (A + 3I)x \) and the image of \( Z = \{[0,0,\gamma]^t|\gamma \in R\} \) under the linear transform \( h(x) = (A + 2I)x \), respectively?

(E) Let \( L : R^2 \to R^2 \) be a linear transform such that \( L([2,1]^t) = [1,0]^t, L([-1,2]^t) = [0,1]^t \), what is \( L([3,4]^t) \)?

(F) Let \( x \in R^2 \), sketch the figure of \( (a) \|x\|_1 = 1 \), \( (b) \|x\|_2 = 1 \), and \( (c) \|x\|_\infty = 1 \).
III. (30%) Answer each of the following questions.

(A) Let \( x, y, z \in \mathbb{R}^5 \) be orthonormal vectors, evaluate \( \|x + 2y + 3z\|_2 \).

(B) Let \( x = [1, 2, 2]^t \), \( y = [1, 0, 1]^t \), find the angle between \( x \) and \( y \).

(C) Let \( V = \{[0, 0, c]^t | c \in \mathbb{R}\} \subset \mathbb{R}^3 \), find \( V^\perp \).
(D) Let \( \mathbf{a} = [1, 1, 1]^t \), \( \mathbf{b} = [1, 3, 8]^t \), find the projection of \( \mathbf{b} \) along the line \( \mathbf{a} \).

(E) Let \( \mathbf{x} = [1, 1, 1, 1]^t \), \( \mathbf{y} = [1, 3, 5, 7]^t \), and \( \mathbf{H} \in \mathbb{R}^{4 \times 4} \) is orthogonal, compute \( \langle \mathbf{Hx}, \mathbf{Hy} \rangle \).

(F) Let \( f, g \in C[-\pi, \pi] \), and define the inner product \( \langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx \), compute \( \langle \sin 3x, \sin 3x \rangle \) and \( \langle \sin 2x, \cos 3x \rangle \), respectively.
IV. (10%) Let $x_1, x_2, \ldots, x_k$ with $k \leq n$ be linearly independent vectors in $\mathbb{R}^n$ and let $A \in \mathbb{R}^{n \times n}$ be a nonsingular matrix. Define $y_j = Ax_j$ for $1 \leq j \leq k$. Prove that $y_1, y_2, \ldots, y_k$ are linearly independent.

V. (10%) Let $u, v \in \mathbb{R}^n$ and $\|u\|_2 = \|v\|_2 = 1$, define $H = I - 2uu^t$.

(a) Show that $H^t = H$ ($H$ is symmetric).

(b) Show that $H^tH = I$ ($H$ is orthogonal).

(c) Show that $\det(H) = -1$. 
VI. (20%) Given \([x_i, y_i]^t \in \mathbb{R}^2\) for \(1 \leq i \leq 5\), the problem of a best fitting line is equivalent to finding \(a\) and \(b\) to minimize

\[
f(a, b) = \sum_{i=1}^{5} (y_i - a \cdot x_i - b)^2
\]

Write Matlab codes to find \(a\) and \(b\).