(20 pts) 1. Mark ☑ if the statement is true, and mark ✗ otherwise, or give your comments.

(☐) (a) Not every underdetermined linear system has a solution.

(☐) (b) Not every nonsingular matrix has an LU-decomposition.

(☐) (c) If $\lambda$ is an eigenvalue of matrix $A$, then $\lambda^m$ must be an eigenvalue of $A^m$.

(☐) (d) $L : R^m \to R^n$ is a linear transform, then $\text{Ker}(L)$ is a vector subspace of $R^m$.

(☐) (e) Let $X, Y$ be 1-dimensional vector subspaces of $R^2$ and $X \perp Y$, then $R^2 = X \oplus Y$.

(✗) (f) The product of eigenvalues of $A$ equals the product of diagonal elements of $A$.

(✗) (g) All eigenvalues of a real symmetric matrix must be distinct.

(✗) (h) Every nonsingular square matrix can be diagonalized.

(✗) (i) Let $A, B \in R^{n\times n}$ be symmetric, then $(A + B)(A - B) = A^2 - B^2$.

(✗) (j) Similar matrices always have the same eigenvectors.

(☐) (k) Let $x \in R^n$ with $\|x\|_2 = 2$. If $A \in R^{n\times n}$ is orthogonal, then $\|Ax\|_2 = 2$. 
Choose the best solution for each of the following questions.

(2) (a) Let \( \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n \) be orthonormal vectors, then \( \|\mathbf{u} - 2\mathbf{v} + 2\mathbf{w}\|_2 = ? \)

(1) 1, (2) 3, (3) 5, (4) 5n, (5) none.

(4) (b) Define \( E(a) = I - ae_2e_1^t \in \mathbb{R}^{n \times n} \) if \( a \neq 0 \), then the inverse matrix of \( E(a) \) is

(1) \( E(a^{-1}) \), (2) \( E(-a^{-1}) \), (3) \( E(a) \), (4) \( E(-a) \), (5) none.

(4) (c) Let \( A \in \mathbb{R}^{n \times n} \) have rank \( r \), then \( \dim(\text{Null}(A)) + \dim(\text{R}(A)) = ? \)

(1) \( m - r \), (2) \( n - r \), (3) \( m \), (4) \( n \), (5) none.

(1) (d) Let \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \), \( \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), the least squares solution of \( Ax = b \) is

(1) \( [1, 1]^t \), (2) \( [1, 0]^t \), (3) \( [0, 1]^t \), (4) \( [-1, -1]^t \), (5) none.

(3) (e) Let \( H_1, H_2, \cdots, H_k \in \mathbb{R}^{n \times n} \) be Householder matrices. Then \( \det(\prod_{j=1}^k H_j) = ? \)

(1) 1, (2) \(-1\), (3) \((-1)^k\), (4) \((-1)^n\), (5) none.

(2) (f) Let \( A \in \mathbb{R}^{n \times n} \) have eigenvalues 1, 3, 5, \( \cdots \), \( 2n - 1 \). Then the trace of \( A \) is

(1) \( n \), (2) \( n^2 \), (3) \( n(n - 1) \), (4) \( n(n + 1) \), (5) none.

(1) (g) Let \( A \in \mathbb{R}^{3 \times 3} \) have \( \lambda(A) = \{1, 2, 5\} \). What is \( \lambda(A^{-1}) \)?

(1) \( \{1, 0.2, 0.5\} \), (2) \( \{-1, -2, -5\} \), (3) \( \{1, 8, 125\} \), (4) \( \{0, 1, 4\} \), (5) none.

(3) (h) Let \( V = \text{Span}(\mathbf{e}_1, \mathbf{e}_3) \leq \mathbb{R}^n \), then \( \dim(V^\perp) = ? \)

(1) 1, (2) 2, (3) \( n - 2 \), (4) \( n - 1 \), (5) none.

(3) (i) Let \( \mathbf{x} = [2, 0, -2]^t \), \( \mathbf{y} = [0, 2, -2]^t \), then the angle between \( \mathbf{x} \) and \( \mathbf{y} \), \( \angle(\mathbf{x}, \mathbf{y}) \) is

(1) \( \frac{\pi}{6} \), (2) \( \frac{\pi}{4} \), (3) \( \frac{\pi}{3} \), (4) \( \frac{\pi}{2} \), (5) none.

(2) (j) Let \( A \in \mathbb{R}^{3 \times 3} \) have eigenvalues 3, 4, 6 what are the eigenvalues of \( (A - 2I)^{-1} \)?

(1) \( \{1, 2, 4\} \), (2) \( \{1, 0.5, 0.25\} \), (3) \( \{2, 0.5, 0.25\} \), (4) \( \{5, 6, 8\} \), (5) none.
(40 pts) 3. Let \( A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \).

(a) Find the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of matrix \( A \) and their corresponding unit eigenvectors \( u_1 \) and \( u_2 \).

(b) Find the trace of \( A \) and the determinant of \( A \).

(c) Let \( U = [u_1, u_2] \), compute \( U^tAU \).

(d) Find the eigenvalues \( \mu_1 \) and \( \mu_2 \) of matrix \( A^{-1} \).

(e) Find the trace of \( A^{-1} \) and the determinant of \( A^{-1} \).

(f) Do the same problems of (a \(\sim\) e) for matrix \( B \).

Ans: \( p(A) = (\lambda + 1)(\lambda + 3) = 0 \).

\( (a) \lambda_1 = -1, \ u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \) and \( \lambda_2 = -3, \ u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \) and

\( (b) \ tr(A) = \lambda_1 + \lambda_2 = -4 \) and \( det(A) = \lambda_1 \times \lambda_2 = 3 \).

\( (c) \ U = [u_1, u_2], \ U^{-1}AU = U^tAU = \text{diag}(-1, -3). \)

\( (d) \mu_1 = -1 \) and \( \mu_2 = -\frac{1}{3} \) for \( A^{-1} \).

\( (e) \ tr(A^{-1}) = -\frac{4}{3} \) and \( det(A^{-1}) = \frac{1}{3}. \)

B-Ans: \( p(B) = (\lambda - 1)^2 = 0 \).

\( (Ba) \lambda_1 = \lambda_2 = 1, \ u_1 = u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \)

\( (Bb) \ tr(B) = \lambda_1 + \lambda_2 = 2 \) and \( det(B) = \lambda_1 \times \lambda_2 = 1 \).

\( (Bc) \ U = [u_1, u_2], \ U^tBU = [1, 1; 1, 1] \) and

\( (Bd) \mu_1 = 1 \) and \( \mu_2 = 1 \) for \( B^{-1} \).

\( (Be) \ tr(B^{-1}) = 2 \) and \( det(B^{-1}) = 1. \)
(30 pts) 4. Let \( A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \), then \( A^2 = \begin{bmatrix} 5 & -4 & 0 \\ -4 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix} \), and define \( \alpha = \min_{\|x\|_2=1} \{ x^t A^5 x \} \), \( \beta = \max_{\|y\|_2=1} \{ y^t A^5 y \} \).

(a) Find the eigenvalues \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) of matrix \( A \) and their corresponding unit eigenvectors \( u_1, u_2, \) and \( u_3. \)

(b) Find the eigenvalues \( \mu_1, \mu_2, \) and \( \mu_3 \) of matrix \( A^2 \) and their corresponding unit eigenvectors \( v_1, v_2, \) and \( v_3. \)

(c) Find the eigenvalues \( \tau_1, \tau_2, \) and \( \tau_3 \) of matrix \( A^5 \) and their corresponding unit eigenvectors \( w_1, w_2, \) and \( w_3. \)

(d) Compute \( \alpha \) and \( \beta. \)

Ans: \( p(A) = (\lambda + 1)(\lambda + 3)(\lambda - 2) = 0. \)

\( (a) \quad \lambda_1 = -1, \ u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \lambda_2 = -3, \ u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \lambda_3 = 2, \ u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \)

\( (b) \quad \mu_1 = (-1)^2 = 1, \ v_1 = u_1, \quad \mu_2 = (-3)^2 = 9, \ v_2 = u_2, \quad \text{and} \quad \mu_3 = 2^2 = 4, \ v_3 = u_3 \text{ for matrix } A^2. \)

\( (c) \quad \tau_1 = (-1)^5 = -1, \ w_1 = u_1, \quad \tau_2 = (-3)^5 = -243, \ w_2 = u_2, \quad \text{and} \quad \tau_3 = 2^5 = 32, \ w_3 = u_3. \)

\( (d) \quad \alpha = (-3)^5 = -243 \text{ and } \beta = (2)^5 = 32. \)
(10 pts) 5. Let $A \in \mathbb{R}^{n \times n}$ and $A^t = A$, show that the eigenvectors corresponding to distinct eigenvalues are orthogonal.

(Proof) Let $\lambda$ and $\mu$ be two distinct eigenvalues of $A$ with corresponding eigenvectors $x$ and $y$, then we have

$$Ax = \lambda x \quad \rightarrow \quad y^t Ax = \lambda y^t x = \lambda \langle x, y \rangle$$

and

$$Ay = \mu x \quad \rightarrow \quad x^t Ay = \mu x^t y = \mu \langle x, y \rangle$$

Since $A^t = A$, then $(y^t Ax)^t = x^t A^t y = x^t Ay$, thus $\lambda \langle x, y \rangle = \mu \langle x, y \rangle$, which implies that $(\lambda - \mu) \langle x, y \rangle = 0$ because $\lambda \neq \mu$, and hence $\langle x, y \rangle = 0$ or say, $x$ and $y$ are orthogonal.
(20 pts) 6. Let \( A = [a_{ij}] \in \mathbb{R}^{m \times n} \), and define \( \|A\|_1 = \text{Max}_{\|u\|_1 = 1} \{\|Au\|_1\} \). Show that

\[
\|A\|_1 = \text{Max}_{1 \leq j \leq n} \left\{ \sum_{i=1}^{m} |a_{ij}| \right\}
\]

(Proof) Let \( \sum_{i=1}^{m} |a_{iK}| = \text{Max}_{1 \leq j \leq n} \{\sum_{i=1}^{m} |a_{ij}|\} \), for any \( x \in \mathbb{R}^n \) with \( \|x\|_1 = 1 \), we have

\[
\|Ax\|_1 = \sum_{i=1}^{m} |\sum_{j=1}^{n} a_{ij} x_j| \\
\leq \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij} x_j| \\
= \sum_{j=1}^{n} |x_j| |\sum_{i=1}^{m} a_{i1} x_j| \\
= \sum_{j=1}^{n} |x_j| \{\sum_{i=1}^{m} |a_{ij}|\} \\
\leq \sum_{j=1}^{n} |x_j| \{\sum_{i=1}^{m} |a_{iK}|\} \\
= \{\sum_{j=1}^{n} |x_j|\} \{\sum_{i=1}^{m} |a_{iK}|\} \\
= \|x\|_1 \{\sum_{i=1}^{m} |a_{iK}|\} \\
= \sum_{i=1}^{m} |a_{iK}|
\]

Thus,

\[
\text{Max}_{\|u\|_1 = 1} \{\|Au\|_1\} \leq \text{Max}_{1 \leq j \leq n} \left\{ \sum_{i=1}^{m} |a_{ij}| \right\} = \sum_{i=1}^{m} |a_{iK}| \text{ for a } K, \ 1 \leq K \leq m.
\]

In particular, when \( x \in \mathbb{R}^n \) is selected as \( x = e_K \), that is, \( x_K = 1 \), and \( x_i = 0 \ \forall \ 1 \leq i \leq n, \ i \neq K \), then the above equality holds, which completes the proof.
(30 pts) 7. Let \( b = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix} \), \( C = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \). Writing a single Matlab command to solve each of the following questions for \( a \sim h \) and answer the questions for \( i \sim h \).

(a) Randomly generate a 3 by 3 matrix \( A \) whose elements are integers in \([0, 10)\). \( A = \text{fix}(10*\text{random}('\text{unif}',0,1,3,3)) \)

(b) Input vector \( b \).
\( (b = [1; -5; 4]) \)

(c) Solve the linear system \( Ax = b \) for \( x \).
\( (x = A\backslash b) \)

(d) Input matrix \( C \) given above.
\( (C = [-2,1,0; 1,-2,0; 0,0,2]) \)

(e) Compute the characteristic polynomial for \( C \). \( p=\text{poly}(C) \)

(f) Compute the eigenvalues and eigenvectors of \( C \). \( [U, D]=\text{eig}(C) \)

(g) Compute the trace of matrix \( C \). \( \text{trace}(C) \)

(h) Compute the rank of matrix \( C \). \( \text{rank}(C) \)

(i) Compute the LU – decomposition of the matrix \( C \). \( [L,U,P]=\text{lu}(C) \)

(j) Compute the QR – factorization of the matrix \( C \). \( [Q, R]=\text{qr}(C) \)

(k) What is the result of (e) ? \( p=[1,2,5,6] \)

(l) What is the result of (f) ? -3, -1, 2, also see problem 4 for the corresponding eigenvectors.

(m) What is the result of (g) ? -2

(n) What is the result of (h) ? 3

(o) What is the result of (i) ?

(p) What is the result of (j) ?