Solutions for Test 3
December 13, 2010

(20%) 1. Mark ☐ if the statement is true, and mark × otherwise.

( x ) (a) A set of nonzero linearly independent vectors must be mutually orthogonal.

( x ) (b) A set of nonzero orthonormal vectors in $\mathbb{R}^n$ must be a basis.

( o ) (c) Every square matrix can be factored as QR, where Q is orthogonal and R is upper-$\Delta$.

( x ) (d) If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{z} \rangle = 0$, then $\langle \mathbf{x}, \mathbf{z} \rangle = 0$.

( x ) (e) If $U, V, W$ are vector subspaces of $\mathbb{R}^n$ such that $U \perp V$ and $V \perp W$, then $U \perp W$.

( o ) (f) Let $U = \text{span}([1, -1]^t)$ and $V = \text{span}([1, 1]^t)$, then $R^2 = U \oplus V$.

( o ) (g) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be linearly independent vectors in $\mathbb{R}^3$, then Gram-Schmidt orthogonalization process can construct an orthonormal basis for $\mathbb{R}^3$.

( o ) (h) Let $H_1, H_2, \cdots, H_k \in \mathbb{R}^{n \times n}$ be Householder matrices, then $\prod_{j=1}^{k} H_j$ is symmetric and orthogonal.

( x ) (i) A Householder matrix is symmetric, orthogonal, and has determinant 1.

( o ) (j) In $\mathbb{R}^n$, if $\mathbf{p}$ is the projection of $\mathbf{b}$ along the line $\mathbf{a}$, then $\langle \mathbf{b} - \mathbf{p}, \mathbf{a} \rangle = 0$.

( x ) (k) If $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^n$, then the vector projection of $\mathbf{x}$ onto $\mathbf{y}$ is equal to the vector projection of $\mathbf{y}$ onto $\mathbf{x}$. 
(40%) 2. Answer the following questions

(A) Let \( u, v, w, z \in \mathbb{R}^n \) be orthonormal vectors, then \( \|u - 3v + 5w - z\|_2 = 6 \)

(B) Let \( x = [1, 2, 1, 2]^t, y = [1, -1, -1, 1]^t \), then the angle between \( x \) and \( y = \frac{\pi}{2} \)

(C) Let \( V = \{[b, 0, a]^t | a, b \in \mathbb{R}\} \subset \mathbb{R}^3 \), then \( V^\perp = \{\beta [0, 1, 0]^t\} \)

(D) Let \( u = [1, 2, 3, 4]^t \), then the rank of \( uu^t = 1 \)

(E) Let \( a = [1, 1, 1]^t, b = [1, 3, 8]^t \), then the projection of \( b \) along the line \( a = 4a \)

(F) Let \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \ b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), the least squares solution of \( Ax = b \) is \( [1, 1]^t \)

(G) The point on the line \( y = 2x + 1 \) that is closest to \( [5, 2]^t \) is \( [1.4, 3.8]^t \)

(H) Let \( x = [1, 1, 1, 1, 1, 1, 1, 1]^t, y = [1, 2, 3, 4, 5, 6, 7, 8]^t \), and \( Q \in \mathbb{R}^{8 \times 8} \) is orthogonal, then \( \langle Qx, Qy \rangle = 36 \)

(I) Let \( y = [-1, 3, -5, 1]^t \), then \( \|y\|_1 + \|y\|_2 + \|y\|_\infty = 21 \)

(J) Let \( f, g \in C[-1, 1], \) and define the inner product \( \langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx \), then \( \langle \sin 2\pi x, \sin 2\pi x \rangle = 1, \langle \cos 4\pi x, \cos 3\pi x \rangle = 0 \).
(10%) 3. Let $x \in \mathbb{R}^n$, show that $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$.

(10%) 4. Let $x, y \in \mathbb{R}^n$, show that $(x - y) \perp (x + y)$ iff $\|x\|_2 = \|y\|_2$.

(20%) 5. Let $u, v \in \mathbb{R}^n$ and $\|u\|_2 = \|v\|_2 = 1$, define $H = I - 2uu^t$.

(a) Show that $H^t = H$ ($H$ is symmetric).

(b) Show that $H^tH = I$ ($H$ is orthogonal).

(c) Show that $det(H) = -1$.

(d) Let $u = [\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}]^t$ and $U = I - 2uu^t$, compute $U$.

(e) Find a Householder matrix $A \in \mathbb{R}^{n \times n}$ such that $Au = v$. 