CS5321 Numerical Optimization Homework 6

Due May 21

1. (40%) Prove the formula of SR1.

   (a) Verify the Sherman-Morrison-Woodbury formula.
   For $\hat{A} = A + \vec{a}\vec{b}^T$,

   $$\hat{A}^{-1} = A^{-1} - \frac{A^{-1}\vec{a}\vec{b}^T A^{-1}}{1 + \vec{b}^T A^{-1}\vec{a}}.$$

   (Hint: prove $\hat{A}\hat{A}^{-1} = \hat{A}^{-1}\hat{A} = I$.)

   (b) Use (a) and the fact that B_k is symmetric to prove that if

   $$B_k = B_{k-1} + \frac{(\vec{y}_k - B_{k-1}\vec{s}_k)(\vec{y}_k - B_{k-1}\vec{s}_k)^T}{(\vec{y}_k - B_{k-1}\vec{s}_k)^T\vec{s}_k},$$

   then

   $$B_k^{-1} = B_{k-1}^{-1} + \frac{(\vec{s}_k - B_{k-1}^{-1}\vec{y}_k)(\vec{s}_k - B_{k-1}^{-1}\vec{y}_k)^T}{\vec{y}_k^T(\vec{s}_k - B_{k-1}^{-1}\vec{y}_k)}.$$

2. (20%) The steepest descent method has search direction $\vec{p}_k = -\vec{g}_k$ and step length $\alpha_k = \dfrac{\vec{g}_k^T\vec{g}_k}{\vec{g}_k^T H_k \vec{g}_k}$. Prove that for a quadratic function

   $$f(\vec{x}) = \frac{1}{2}\vec{x}^T H\vec{x} + \vec{g}^T\vec{x} + f,$$

   $\vec{p}_k \perp \vec{p}_{k+1}$.

3. (40%) In the class, we had shown that in CG,

$$\alpha_k = \frac{-\vec{p}_k^T \vec{g}_k}{\vec{p}_k^T H \vec{p}_k} \text{ and } \beta_k = \frac{\vec{p}_{k-1}^T H \vec{g}_k}{\vec{p}_{k-1}^T H \vec{p}_{k-1}}.$$

But in the handout, it says

$$\alpha_k = \frac{\vec{g}_k^T \vec{g}_k}{\vec{p}_k^T H \vec{p}_k} \text{ and } \beta_k = \frac{\vec{g}_k^T \vec{g}_k}{\vec{g}_{k-1}^T \vec{g}_{k-1}}$$

To bridge this gap, let's prove the following equalities

(a) $\vec{g}_k^T \vec{p}_{k-1} = 0$

(b) $\vec{g}_k^T \vec{g}_k = -\vec{g}_k^T \vec{p}_k$

(c) Prove those two relations together (Hint: by induction!!)

$$\vec{g}_k^T \vec{p}_j = 0 \text{ for } k > j,$$

$$\vec{p}_k^T H \vec{p}_j = 0 \text{ for } k \neq j.$$

(d) Prove $\beta_{k+1} = \dfrac{\vec{g}_k^T \vec{g}_k}{\vec{g}_{k-1}^T \vec{g}_{k-1}}$. (You may use the result of (c) directly even if you cannot prove it.)