CS5321 Numerical Optimization Homework 5

Due May 9

1. (30%) Use the optimality conditions of constrained optimization problems to verify the following properties.

   (a) The optimal solution of the total least square problem is $A^T A\vec{x} = \lambda\vec{x}$ for some λ.

   (b) For the trust region method, the optimal solution $\vec{p}^*$ of the local model

   $$\min_{\vec{p}\in\mathbb{R}^n} m(\vec{p}) = \vec{g}^T\vec{p} + \frac{1}{2}\vec{p}^T A\vec{p} \text{ s.t. } \vec{p}^T\vec{p} \le \Delta^2,$$

   satisfies

   $$(A+\lambda I)\vec{p}^* = -\vec{g}, \;\; \lambda(\Delta - \|\vec{p}^*\|) = 0, \text{ and } (A+\lambda I) \text{ is positive definite.}$$

2. (30%) For a quadratic programming,

   $$\min_{\vec{x}} g(\vec{x}) = \frac{1}{2}\vec{x}^T G\vec{x} + \vec{x}^T\vec{c}$$

   $$\text{s.t. } A\vec{x} = \vec{b},$$

   Prove that if A has full row-rank and the reduced Hessian $Z^T G Z$ is positive definite, where span$\{Z\}$ is the null space of span$\{A^T\}$, then the KKT matrix $K = \begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix}$ is nonsingular. (Hint: Prove that every vector $\begin{bmatrix} \vec{w} \\ \vec{v} \end{bmatrix}$ making $\begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} \vec{w} \\ \vec{v} \end{bmatrix} = 0$ is a zero vector. Using the property that $\vec{w}^T G\vec{w} > 0$.)

3. (40%) Consider the quadratic programming problem with bounded constraints

   $$\min_{x_1,x_2,x_3} (x_1-4)^2 + (x_2-3)^2 + (x_3-2)^2$$

   $$\text{s.t. } 0 \le x_1, x_2, x_3 \le 2$$

   Use gradient projection method to find its optimal solution with $\vec{x}_0 = 0$. Write down the trace, like

   $$\vec{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to \vec{x}_1 = \begin{pmatrix} 2 \\ 3/2 \\ 1 \end{pmatrix} \to \vec{x}_2 = \cdots$$