CS5321 Numerical Optimization Homework 2

Due March 24

1. (20%) A set $\mathcal{S}$ convex if the straight line connecting any two points in $\mathcal{S}$ is entirely in $\mathcal{S}$. A function is called *convex* if its domain $\mathcal{S}$ is convex, and for any $\vec{x}, \vec{y} \in \mathcal{S}$,

$$f(\alpha\vec{x} + (1-\alpha)\vec{y}) \leq \alpha f(\vec{x}) + (1-\alpha) f(\vec{y}),$$

for all $\alpha \in [0,1]$.

   (a) Prove that when f is convex, any local minimizer $\vec{x}^*$ is a global minimizer of f. (Hint: Suppose there is another point $\vec{z} \in \mathcal{S}$ such that $f(\vec{z}) \leq f(\vec{x}^*)$. Then $\vec{x}^*$ is not a local minimizer.)

   (b) Suppose $f(\vec{x}) = \vec{x}^T Q \vec{x}$, where Q is a symmetric positive semidefinite matrix. Show that $f(\vec{x})$ is convex. (Hint: It might be easier to show $f(\vec{y} + \alpha(\vec{x} - \vec{y})) - \alpha f(\vec{x}) - (1-\alpha) f(\vec{y}) \leq 0$.)

2. (30%) For a given function $f(x) : \mathbb{R} \to \mathbb{R}$,

   (a) What is the quadratic polynomial $p(x)$ satisfying $p(0) = f(0)$, $p(1) = f(1)$, and $p'(0) = f'(0)$? Express $p(x)$ by $f(0)$, $f(1)$, and $f'(0)$.

   (b) What is the minimizer of $p(x)$ for $x \in [0,1]$? You may need to discuss different cases for different $f(0)$, $f(1)$, and $f'(0)$.

   (c) What is the cubic polynomial $q(x)$ satisfying $q(0) = f(0)$, $q(\alpha_1) = f(\alpha_1)$, $q(\alpha_2) = f(\alpha_2)$, and $q'(0) = f'(0)$? Express $q(x)$ by $f(0)$, $f(\alpha_1)$, $f(\alpha_2)$, and $f'(0)$.

3. (50%) Let $f_1(x,y) = \frac{1}{2}x^2 + \frac{9}{2}y^2$ and $f_2(x,y) = \frac{1}{2}x^2 + y^2$.

   (a) Derive the gradient g and the Hessian H of f_1 and f_2, and compute Hs' eigenvalues.

   (b) Write Matlab codes to implement the steepest descent method and Newton's method with $\vec{x}_0 = (9,1)$, and compare their convergent results. The formula of the steepest descent method is

$$\vec{x}_{k+1} = \vec{x}_k - \frac{\vec{g}_k^T \vec{g}_k}{\vec{g}_k^T H_k \vec{g}_k} \vec{g}_k,$$

   and the formula of Newton's method is

$$\vec{x}_{k+1} = \vec{x}_k - H_k^{-1} \vec{g}_k,$$

   where $\vec{g}_k = g(\vec{x}_k)$ and $H_k = H(\vec{x}_k)$.