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Lecture Notes 7: Eigenvalue Problem

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3 Eigenvalue Problem

3.1 QR Algorithm

- Want to compute all eigenpairs of A .
- Algorithm
  1. B_0=A,U_0=I
  2. For i=0, 1, 2...until converge ...Note(1)
  3. $[Q_{i+1}, R_{i+1}]$=qr(B_i)
  4. B_{i+1}=$R_{i+1} * Q_{i+1}$, $U_{i+1} = U_i * Q_i$
  5. end For
  6. Ectract eigenpairs from U_k,B_k

  Extract A's engenpairs

  1. Compute B_k's eigenpairs
  2. B_k=$Y_k \Lambda_k Y_k^{-1}$ // diag(Λ) are A's eigenvalue.
  3. A's eigrnmatrix = $U_k * Y_k$

  Note(1): B_k will converge to a block upper triangular matrix

$$
\begin{aligned}
B_{k+1} &= R_k * Q_k \qquad //by\ operation,\ R_k = Q_k^T * B_k \\
&= Q_k^T B_k Q_k \\
&= Q_k^T (Q_{k-1}^T B_{k-1} Q_{k-1}) Q_k \qquad //recursively \\
&= \dots \\
&= Q_k^T Q_{k-1}^T \dots Q_1^T Q_0^T B_0 Q_0 Q_1 \dots Q_{k-1} Q_k \\
&= U_{k+1} A U_{k+1} \qquad //This\ is\ an\ upper\ triangular\ matrix.
\end{aligned}
$$

- Theorem: $U_k = Z_k$, where U_k is from QR algorithm and Z_k is from OI algorithm. pf:(Use Mathematical Induction)

$$U_0 = Z_0$$

  Assume iteration holds.

$$Z_k = U_k = Q_1 Q_2 \dots Q_{k-1}$$

In the orthogonal iteration,

$$Y_{k+1} = AZ_k$$
$$Z_{k+1}W_{k+1} = Y_{k+1} = AZ_k$$

Then,

$$\begin{aligned} W_{k+1} &= Z_{k+1}^T AZ_k \\ &= Z+k+1^T Z_k Z_k^T AZ_k \qquad //Rayleigh\ \ Quotient:\ \ Z_k^T AZ_k \\ &= Z+k+1^T Z_k U_k^T AU_k \\ &= Z_{k+1}^T Z_k B_k \qquad //from\ \ Note(1) \end{aligned}$$

$$\begin{aligned} B_k &= (Z_{k+1}^T Z_k)^T W_{k+1} \\ &= Z_k^T Z_{k+1} W_{k+1} \\ &= Q_{k+1}R_{k+1} \end{aligned}$$

Thus,

$$\begin{aligned} Z_k^T Z_{k+1} &= Q_{k+1} \\ Q_k^T \dots Q_1^T Z_{k+1} &= Q_{k+1} \end{aligned}$$

$$\begin{aligned} Z_{k+1} &= Q_1 Q_2 \dots Q_k Q_{k+1} \\ &= U_{k+1} \end{aligned}$$

□

- How to make A become an upper triangular?
  Assume A is a 5X5 matrix.

$$A = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$

Givens Rotation,

$$Q_1 = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & c & -s \\ & & & s & c \end{bmatrix} \quad Q_1^T = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & c & s \\ & & & -s & c \end{bmatrix}$$

Then,

$$Q_1 A Q_1^T = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \end{bmatrix}$$

$$Q_3 Q_2 Q_1 A Q_1^T Q_2^T Q_3^T = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \end{bmatrix}$$

$$Q_4 Q_3 Q_2 Q_1 A Q_1^T Q_2^T Q_3^T Q_4^T = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \end{bmatrix}$$

$$Q_6 Q_5 Q_4 Q_3 Q_2 Q_1 A Q_1^T Q_2^T Q_3^T Q_4^T Q_5^T Q_6^T = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix} \qquad //upper\ \ Hessenberg$$

- Algorithm(use upper Hessenberg)
  1. $B_0 = Q_0 A Q_0^T$ //Let B_0 be an upper Hessenberg.
  2. For i=0, 1, 2,... until converge
  3. $[Q_{i+1}, R_{i+1}]$=qr(B_i)
  4. B_{i+1}=$R_{i+1} * Q_{i+1}$, $U_{i+1} = U_i * Q_i$
  5. end For
  6. Ectract eigenpairs from U_k,B_k
- The time complexity of QR algorithm is O(n^3) but we can reduce the time complexity by using upper Hessenberg. And the time complexity of QR algorithm using upper Hessenberg is O(n^2).

3.2 Eigen Decompositions for Symmetric Matrices

- If A is symmetric, then
  1. All its eigenvalues are real.
     pf:
     $$\begin{aligned} A &= \bar{A}^T \\ &= \bar{Q}T^T\bar{Q}^T \\ \Rightarrow \bar{Q} &= Q, \bar{T}^T = T \end{aligned}$$
     Thus, Q and T are real matrices. □
  2. All its eigenvectors can be orthogonal.
     pf:
     $$\begin{aligned} A &= QTQ^T \\ A^T &= QT^TQ^T \\ A &= A^T \\ \Rightarrow T &= T^T \end{aligned}$$
     Thus, T is diagonal. □

- Define: Interial(A)=(α, β, γ)

  α: # of positive eigenvalues of A.

  β: # of zero eigenvalues of A.

  γ: # of negative eigenvalues of A.

- Example:

  Inertial($\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$)=(1, 1, 0)

- Theorem: If Y is nonsingular and A is symmetric, then
  $$\begin{aligned} Inertial(A) &= Inertial(T^TAY) \\ (\alpha_1, \beta_1, \gamma_1) &= (\alpha_2, \beta_2, \gamma_2) \end{aligned}$$
  Note that:

  The values of diagonal of Y^TAY does not always equal to the eigenvalues of A.