Advanced Numerical Methods	Nov 30, and Dec 2, 2011

Lecture Notes 7: Eigenvalue Problem

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3 Eigenvalue Problem

3.1 Orthogonal iteration

- Want to compute more than one eigenpairs at a time.
- Algorithm(OI)
  1. Z_0=$[\vec{v}_1, \vec{v}_2]$ is an nx2 random orthogonal matrix.
  2. For i=1, 2, 3...
  3. Y_i=A*Z_{i-1}
  4. $[Z_i, W_i]$=QR(Y)
  5. end for
  6. Extract eigenpairs from Z.
     (a) T_k=Z_k^TAZ_k
     (b) if T_k is upper triangular converged($Z_k = Q_k$)
     (c) compute T_k's eigenpairs.
- Suppose $X = [\vec{x}_1, \vec{x}_2, ...\vec{x}_n]$ eigenvector matrix of A.

$$
\begin{aligned}
Assume \quad & |\lambda_1| > |\lambda_2| > \ldots |\lambda_n| \\
& X = QR \\
& X^{-1} = R^{-1}Q^T \\
& A = X\Lambda X^{-1} = QR\Lambda R^{-1}Q^T = QTQ^T (Schur \quad decomposition) \\
Assume \quad & X = [x_1 x_2] = [Q_1 Q_2] \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} \\
& X^{-1} = \begin{bmatrix} R_{11}^{-1} & R_{11}^{-1} R_{12} R_{22}^{-1} \\ 0 & R_{22}^{-1} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
Q_1^T A Q_1 &= Q_1^T X \Lambda X^{-1} Q_1 \\
&= Q_1^T [Q_1 Q_2] \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} \begin{bmatrix} \Lambda_1 & \\ & \Lambda_2 \end{bmatrix} \begin{bmatrix} R_{11}^{-1} & R_{11}^{-1} R_{12} R_{22}^{-1} \\ 0 & R_{22}^{-1} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} Q_1 \\
&= [R_{11} R_{12}] \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} R_{11}^{-1} \\ 0 \end{bmatrix} \\
&= R_{11} \Lambda_1 R_{11}^{-1}
\end{aligned}
$$

- eigenvalues and eigenvectors of an upper triangular matrix T

$$T = \begin{bmatrix} t_{11} & t_{12}... & t_{1n} \\ & t_{22}... & t_{2n} \\ & \ddots & \\ & & t_{nn} \end{bmatrix} = \begin{bmatrix} T_{11} & \vec{t}_{1i} & T_{1,i+1} \\ & t_{ii} & \vec{t}^T_{i,i+1} \\ & & T_{i+1,i+1} \end{bmatrix}$$

$$\lambda_i = t_{ii}$$

$$\vec{x}_i = \begin{bmatrix} (t_{ii}I - T_{11})^{-1} & \vec{t}_{1,n} \\ 1 & \\ 0 & \end{bmatrix}$$

- span { Z_k } $\rightarrow$ an invariant subspace of S={$\vec{u}$}

$$\vec{u} \in s \quad A\vec{u} \in s$$

$$A\vec{x}_1 = \lambda_1 \vec{x}_1$$
$$A\vec{x}_2 = \lambda_2 \vec{x}_2$$
$$\vec{u} = \alpha\vec{x}_1 + \beta\vec{x}_2$$
$$A\vec{u} = \alpha\lambda_1\vec{x}_1 + \beta\lambda_2\vec{x}_2$$

3.2 QR Algorithm

- Algorithm
- To Compute all eigenpairs.

1. B_0=A
2. For i=0,1,2...
3. $[Q_{i+1}, R_{i+1}]$=qr(B_i)
4. $B_{i+1} = R_{i+1}Q_{i+1}, U_{i+1} = U_iQ_i$
   end for
5. Extract eigenpairs from U_K, B_K

$$
\begin{aligned}
A^K Z_0 &= X \Lambda^K X^{-1} Z_0 \\
&= X \begin{bmatrix} \lambda_1^k & & & & & & \\ & \lambda_2^k & & & & & \\ & & \ddots & & & & \\ & & & \lambda_p^k & & & \\ & & & & \lambda_{p+1}^k & & \\ & & & & & \ddots & \\ & & & & & & \lambda_n^k \end{bmatrix} X^{-1} Z_0 \\
&= \lambda_p^k X \begin{bmatrix} (\frac{\lambda_1^k}{\lambda_p})^k & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & (\frac{\lambda_{p+1}}{\lambda_p})^k & & \\ & & & & \ddots & \\ & & & & & (\frac{\lambda_n}{\lambda_p})^k \end{bmatrix} X^{-1} Z_0 \\
&\rightarrow \lambda_p^k X \left[X_p X_p^{\perp} \right] \begin{bmatrix} x & & & & & & \\ & x & & & & & \\ & & x & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{bmatrix} X^{-1} Z_0 \\
&\rightarrow \lambda_p^k (X_p W + X_p^{\perp} 0) \\
& [\lambda, \vec{x}_1] \quad converge \quad rate(\frac{\lambda_p}{\lambda_1})
\end{aligned}
$$

− converge rate

$$\lim_{k\to\infty} = \frac{||u_1^{(k)} - \vec{x}_1||}{||\vec{u}_1^{k+1} - \vec{x}_1||} = p = \frac{1}{2}(converge \quad rate) \tag{1}$$

$$\lim_{k\to\infty} = \frac{|u_1^k - \lambda_1|}{|u_1^{k-1} - \lambda_1|} \tag{2}$$

$$\lim_{k\to\infty} = \frac{||A\vec{u}_1^{(k)} - u_1^{(k)}\vec{u}_1^{(k)}|| \rightarrow residual}{||A\vec{u}_1^{(k-1)} - u_1^{(k-1)}\vec{u}_1^{(k-1)}||} \tag{3}$$