CS5321 Numerical Optimization Homework 6

Due May 24

1. (20%) (Farkas's Lemma) Let A be an $m \times n$ matrix and $\vec{b}$ an m vector. Prove that exact one of the following two statements is true:

   (a) There exists a $\vec{v} \in \mathbb{R}^n$ such that $A\vec{v} = \vec{b}$ and $\vec{v} \geq 0$.

   (b) There exists a $\vec{u} \in \mathbb{R}^m$ such that $A^T\vec{u} \geq 0$ and $\vec{b}^T\vec{u} < 0$.

   (Hint: prove if (a) is true, then (b) cannot be true, and vice versa.)

> Farkas's Lemma (1902) plays an important role in the proof of the KKT condition. The most critical part in the proof of the KKT condition is to show that the Lagrange multiplier $\vec{\lambda}^* \geq 0$ for inequality constraints. We can say if the LICQ condition is satisfied at $\vec{x}^*$, then any feasible direction $\vec{y}$ at $\vec{x}^*$ must have the following properties:
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> - $\vec{y}^T \nabla f(\vec{x}^*) \geq 0$, since $\vec{x}^*$ is a local minimizer. (Otherwise, we find a feasible descent direction that decreases f.)
> - $\vec{y}^T \nabla c_i(\vec{x}^*) = 0$ for equality constraints, $c_i = 0$.
> - $\vec{y}^T \nabla c_i(\vec{x}^*) \geq 0$ for inequality constraints, $c_i \geq 0$.
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> Here is how Farkas Lemma enters the theme. Let $\vec{b}$ be $\nabla f(\vec{x}^*)$, $\vec{u}$ be $\vec{y}$ (any feasible direction at $\vec{x}^*$), the columns of A be $\nabla c_i(\vec{x}^*)$. Since no such $\vec{u}$ exists, according to the properties of $\vec{y}$, statement (a) must hold. The vector $\vec{v}$ in (a) corresponds to $\vec{\lambda}^*$, which just gives us the desired result of the KKT condition.

2. (50%) Consider the following constrained minimization problem

$$\min_{x_1,x_2} -x_1 + x_2^2 \text{ subject to } \begin{cases} (1-x_1)^3 - x_2 & \geq 0 \\ x_1 + x_2 - 1 & \geq 0 \end{cases}$$

(a) Plot the feasible region of the problem, and use it to find the optimal solution $\vec{x}^*$.

(b) Write its Lagrangian function $\mathcal{L}(\vec{x}, \vec{\lambda})$, and compute $\vec{\lambda}^*$.

(c) Verify the LICQ condition at $\vec{x}^*$.

(d) Verify the KKT condition at $\vec{x}^*$.

(e) Compute the Lagrangian Hessian at $\vec{x}^*$ and the critical cone, and verify the second order optimality condition.

3. (30%) Consider the following problem

$$\min_{x_1,x_2} \frac{1}{2}\alpha x_1^2 + \frac{1}{2}x_2^2 + x_1 \text{ subject to } x_1 \geq 1.$$

Determine the solution to this problem for $\alpha = 1$ and $\alpha = 0$. For each case, formulate the dual, and determine whether the primal and the dual have the same optimal solution.