CS5321 Numerical Optimization Homework 5

Due May 10

1. (20%) Linear equality constraints.

   (a) Reduce the following problem to an unconstrained problem,

$$\begin{aligned} \min_{x_1,\ldots,x_6} \quad & \sin(x_3 + x_6) + x_1^3 + x_2^2 + x_1x_3x_6 + x_4x_5^2 \\ \text{subject to} \quad & 8x_1 - 6x_2 + x_3 + 9x_4 + 4x_5 = 6 \\ & 3x_1 + 2x_2 - 4x_4 + 6x_5 + 4x_6 = -4 \end{aligned}$$

   (b) Consider the general form of a constrained minimization problem with only linear equality constraints.

$$\min_{\vec{x}} f(\vec{x}) \text{ subject to } A\vec{x} = \vec{b},$$

   where $A \in \mathbb{R}^{m\times n}$ with $m < n$. Suppose A has full row rank. Prove that this problem can be reduced to an unconstrained problem with $n-m$ unknowns. (Hint: use similar technique in the simplex method for representing basic variables by nonbasic variables.)

2. (80%) Consider the following linear programming problem

$$\begin{aligned} \max_{x_1,x_2} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \le 4 \\ & 4x_1 + 2x_2 \le 12 \\ & -x_1 + x_2 \le 1 \\ & x_1, x_2 \ge 0 \end{aligned}$$

   (a) Draw the figure of the constraints and use that to solve the problem.

   (b) Derive its dual problem and solve the dual problem by any means. Compare the solutions of the primal and the dual problems.

   (c) Verify the complementarity slackness condition.

(d) Transform the problem to the standard form.

(e) Solve it by the simplex method, as provided in Figure 1, using $\vec{x}_0 = (0, 0)$. Indicate $B_k, N_k, \vec{s}_k, \vec{d}_k, p_k, q_k$ and γ_k in each step.

(f) Use Matlab function `linprog` to solve the problem. The default method used by `linprog` is the interior point method. How to change it to the simplex method?

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(1) Given a basic feasible point $\vec{x}_0$ and the corresponding index set $\mathcal{B}_0$ and $\mathcal{N}_0$.

(2) For $k = 0, 1, \ldots$

(3) Let $B_k = A(:, \mathcal{B}_k), N_k = A(:, \mathcal{N}_k)$, $\vec{x}_B = \vec{x}_k(\mathcal{B}_k), \vec{x}_N = \vec{x}_k(\mathcal{N}_k)$, and $\vec{c}_B = \vec{c}_k(\mathcal{B}_k), \vec{c}_N = \vec{c}_k(\mathcal{N}_k)$.

(4) Compute $\vec{s}_k = \vec{c}_N - N_k^T (B_k^{-1})^T \vec{c}_B$ (pricing)

(5) If $\vec{s}_k \geq 0$, return the solution $\vec{x}_k$. (found optimal solution)

(6) Select $q_k \in \mathcal{N}_k$ such that $\vec{s}_k(i_q) < 0$,
where i_q is the index of q_k in $\mathcal{N}_k$

(7) Compute $\vec{d}_k = B_k^{-1} A_k(:, q_k)$. (search direction)

(8) If $\vec{d}_k \leq 0$, return `unbounded`. (unbounded case)

(9) Compute $[\gamma_k, i_p] = \min_{i, \vec{d}_k(i) > 0} \frac{\vec{x}_B(i)}{\vec{d}_k(i)}$ (ratio test)
(The first return value is the minimum ratio;
the second return value is the index of the minimum ratio.)

(10) $x_{k+1} \begin{pmatrix} \mathcal{B} \\ \mathcal{N} \end{pmatrix} = \begin{pmatrix} \vec{x}_B \\ \vec{x}_N \end{pmatrix} + \gamma_k \begin{pmatrix} -\vec{d}_k \\ \vec{e}_{i_q} \end{pmatrix}$
($\vec{e}_{i_q} = (0, \ldots, 1, \ldots, 0)^T$ is a unit vector with i_qth element 1.)

(11) Let the i_pth element in $\mathcal{B}$ be p_k. (pivoting)
$\mathcal{B}_{k+1} = (\mathcal{B}_k - \{p_k\}) \cup \{q_k\}$, $\mathcal{N}_{k+1} = (\mathcal{N}_k - \{q_k\}) \cup \{p_k\}$

Figure 1: The simplex method for solving (minimization) linear programming