CS5321
Numerical Optimization

16 Quadratic Programming

Quadratic programming

- Quadratic object function + linear constraints

$$\begin{array}{lll} \min_x & q(x) = \frac{1}{2}x^T G x + x^T c & \\ \text{s.t.} & a_i^T x = b_i & i \in \mathbf{E} \\ & a_i^T x \geq b_i & i \in \mathbf{I} \end{array}$$

  - If G is positive semidefinite, it is called *convex* QP.
  - If G is positive definite, it is called *strictly convex* QP.
  - If G is indefinite, it is called *nonconvex* QP.

Equality constraints

- First order condition for optimality (chap12)

$$\begin{pmatrix} G & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} -p \\ \lambda^* \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix}$$

  - where $h=Ax-b$, $g=c+Gx$, and $p=x^*-x$. Pair (x^*, λ^*) is the optimal solution and Lagrangian multipliers.

- $K = \begin{pmatrix} G & A^T \\ A & 0 \end{pmatrix}$ is called the KKT matrix, which is indefinite, nonsingular if the project Hessian Z^TGZ is positive definite. (chap 12)

Solving the KKT system

- Direct methods
  - Symmetric indefinite factorization $P^{T}KP=LBL^{T}$.
  - Schur complement method (1)
  - Null-Space method (2)
- Iterative methods
  - CG applied to the reduced system (3)
  - The projected CG method (4)
  - Newton-Krylov method

1.Schur-complement method

- Assume G is invertible.

$$\begin{pmatrix} I & 0 \\ AG^{-1} & -I \end{pmatrix} \begin{pmatrix} G & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} -p \\ \lambda^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ AG^{-1} & -I \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix}$$

$$\begin{pmatrix} G & A^T \\ 0 & AG^{-1}A^T \end{pmatrix} \begin{pmatrix} -p \\ \lambda^* \end{pmatrix} = \begin{pmatrix} g \\ AG^{-1}g - h \end{pmatrix}$$

- This gives two equations
  - Solve for λ^* first.
    Then use λ^* to solve p.

$$\begin{aligned} -Gp + A^T\lambda^* &= g \\ (AG^{-1}A^T)\lambda^* &= AG^{-1}g - h \end{aligned}$$

  - Matrix $AG^{-1}A^{\mathrm{T}}$ is called the Schur-complement

2.Null space method

- Require (1) A has full row rank (2) $Z^{T}GZ$ is positive definite. (G itself can be singular.)
- QR decomposition of A^{T}, and partition p (Q_2=Z)

$$A^T = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \begin{pmatrix} R \\ 0 \end{pmatrix}, p = Q_1 p_1 + Q_2 p_2$$

$$\begin{pmatrix} G & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} -p \\ \lambda^* \end{pmatrix} = \begin{pmatrix} G & Q_1 R \\ R^T Q_1^T & 0 \end{pmatrix} \begin{pmatrix} -p \\ \lambda^* \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix}$$

$$\begin{aligned} Rp_1 &= -h \\ (Q_2^T G Q_2) p_2 &= -Q_2^T G Q_1 p_1 - Q_2^T g \\ R^T \lambda^* &= Q_1^T (g + Gp) \end{aligned}$$

3.Conjugate Gradient for $Z^T GZ$

- Assume $A^T = Q_1 R$, $Q_2 = Z$. (see previous slide).
  - The solution can be expressed as $x = Q_1 x_1 + Q_2 x_2$.
  - $x_1 = R^{-T} b$ cannot be changed. Reduce the original problem to the unconstrained one (see chap 15)

$$\min_{x_2} \frac{1}{2} x_2^T Q_2^T G Q_2 x_2 + x_2^T Q_2^T c$$

- The minimizer is the solution of (see chap 2)

$$Q_2^T G Q_2 x_2 = -Q_2^T c$$

  - If $Q_2^T G Q_2$ is s.p.d., the CG can be used as a solver. (see chap 5)

4.The Projected CG method

- Use the same assumptions as the previous slide
- Define $P=Q_2Q_2^T$, an orthogonal projector, by which $Px \in \text{span}(Q_2)$ for all x.
- The projected CG method uses *projected residual* $r_{k+1}^p = Pr_{k+1}$ (see chap 5) instead of r_{k+1}.
- Matrix P is equivalent to $P = I - A^T(AA^T)^{-1}A$.
  - Numerically unstable, use *iterative refinement* to improve the accuracy.

Inequality constraints

- Review the optimality conditions (chap 12)

$$\begin{aligned} Gx^* + c - \sum_{i \in \mathbf{A}(x^*)} \lambda_i^* a_i &= 0 \\ a_i^T x^* &= b_i \quad i \in \mathbf{E} \\ a_i^T x^* &\geq b_i \quad i \in \mathbf{I} \backslash \mathbf{A}(x^*) \\ \lambda^* &\geq 0 \quad i \in \mathbf{I} \cap \mathbf{A}(x^*) \end{aligned}$$

- Two difficulties (Figure 16.1, 16.2)
  - Nonconvexity: more than one minimizer
  - Degeneracy: The gradients of active constraints are linearly dependent

Methods for QP

1. Active set method for convex QP
   - Only for small or medium sized problems
2. Interior point method
   - Good for large problems
3. Gradient projection method
   - Good when the constraints are bounds for variables

1.Active set method for QP

- Basic strategy
  1. Find an active set (called working set, denoted $\mathbf{W}_k$)
  2. Check if x_k is optimal
  3. If not, find p_k (solving an equality constrained QP)

$$\begin{aligned} \min_p \quad & \tfrac{1}{2} p^T G p + g^T p \\ \text{s.t.} \quad & a_i^T p = 0 \qquad\qquad i \in \mathbf{W}_k \end{aligned}$$

  4. Find $\alpha_k \in [0,1]$ such that $x_{k+1} = x_k + \alpha_k p_k$ satisfies all constraints a_i, $i \notin \mathbf{W}_k$ (next slide)
  5. If $\alpha_k < 1$, update $\mathbf{W}_k$.

Step length

- Need find $\alpha_k \in [0,1]$ as large as possible such that $a_i^T(x_k+\alpha_k p_k) \geq b_i,\ i \notin \mathbf{W}_k$
  - If $a_i^T p_k \geq 0$, then for all $\alpha_k \geq 0$, it satisfies anyway
  - If $a_i^T p_k < 0$, α_k need be $\leq (b_i - a_i^T x_k)/a_i^T p_k$.

$$\alpha_k = \min\left(1, \min_{i \notin \mathbf{W}_k, a_i^T p_k < 0} \frac{b_i - a_i^T x_k}{a_i^T p_k}\right)$$

2.Interior point method for QP

- Please review the IPM for LP (chap 14)
- The problem

$$\begin{array}{ll} \min_x & q(x) = \frac{1}{2}x^T G x + x^T c \\ \text{s.t.} & Ax \geq b \end{array}$$

- KKT conditions

$$\begin{array}{rcl} Gx - A^T\lambda + c & = & 0 \\ Ax - y - b & = & 0 \\ y_i\lambda_i & = & 0, i = 1, 2, \ldots, m \\ y, \lambda & \geq & 0 \end{array}$$

- Complementary measure:

$$\mu = y^T\lambda/m$$

- Nonlinear equation

$$F(x, y, \lambda, \sigma\mu) = \begin{pmatrix} Gx - A^T\lambda + c \\ Ax - y - b \\ Y\Lambda e - \sigma\mu e \end{pmatrix}$$

- Using Newton's method to solve $F(x,y,\lambda,\sigma\mu)=0$

$$\begin{pmatrix} G & 0 & -A^T \\ A & -I & 0 \\ 0 & \Lambda & Y \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta\lambda \end{pmatrix} = \begin{pmatrix} -r_d \\ -r_p \\ -\Lambda Y e + \sigma\mu e \end{pmatrix}$$

where

$$\begin{aligned} r_d &= Gx - A^T\lambda + c \\ r_p &= Ax - y - b \end{aligned}$$

- Next step

$$(x_{k+1}, y_{k+1}, \lambda_{k+1}) = (x_k, y_k, \lambda_k) + \alpha_k(\Delta x, \Delta y, \Delta\lambda)$$

3.Gradient projection method

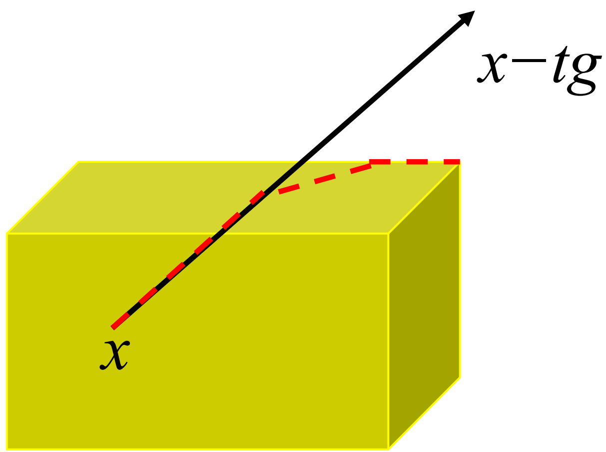
- Consider the bounded constrained problem

$$\begin{array}{ll} \min_x & q(x) = \frac{1}{2}x^T G x + x^T c \\ \text{s.t.} & l \le x \le u, \end{array}$$

  - The gradient of $q(x)$ is $g = Gx+c$.

- Bounded projection

$$P(x,l,u) = \begin{cases} l & \text{if } x < l \\ x & \text{if } l < x < u \\ u & \text{if } x > u \end{cases}$$



- The gradient projection is a piecewise linear path

$$x(t)=P(x-tg,\ l.\ u)$$

- Piecewise linear path

$$x(t) = \begin{cases} x_1 + \Delta t p_1 & \text{for } 0 \le \Delta t < t_1 \quad (x_1 = x) \\ x_2 + \Delta t p_2 & \text{for } t_1 \le \Delta t < t_2 \quad (x_2 = x_1 + t_1 p_1) \\ \vdots & \\ x_n + \Delta t p_n & \text{for } t_{n-1} \le \Delta t \le t \quad (x_n = x_{n-1} + t_{n-1} p_{n-1}) \end{cases}$$

- For each dimension

$$t_i = \begin{cases} (x_i - u_i)/g_i & \text{if } g_i < 0 \\ (x_i - l_i)/g_i & \text{if } g_i > 0 \\ \infty & \text{otherwise} \end{cases}$$

$$x_i(t) = \begin{cases} x_i - t g_i & \text{if } t \le t_i \\ x_i - t_i g_i & \text{otherwise} \end{cases}$$

$$p_i(t) = \begin{cases} -g_i & \text{if } t_{i-1} \le t_i \\ 0 & \text{otherwise} \end{cases}$$