CS5321
Numerical Optimization

14 Linear Programming:
The Interior Point Method

Interior point methods for LP

- There are many variations of IPM for LP
  - Primal IPM, dual IPM, primal-dual IPM
  - Potential Reduction Methods
  - Path Following Methods
    - Short-step methods
    - Long-step methods
    - Predictor-corrector methods
- We will cover the path following method and the long-step method

The dual problem of LP

$$\min_x z = c^T x \text{ subject to } Ax = b,\ x \geq 0$$

- The dual problem is (see chap 12)

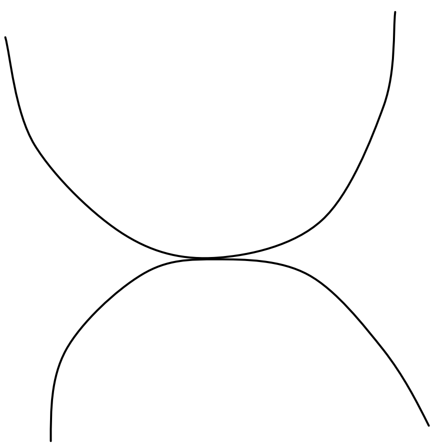
$$\max_\lambda b^T\lambda \quad \text{subject to} \quad A^T\lambda + s = c,\ s \geq 0$$

- Weak duality

  For any x and λ that are feasible in the primal and the dual problems, $b^T\lambda \leq c^T x$.

- Strong duality

  If x^* and λ^* are solutions to the primal and the dual problems, $b^T\lambda^* = c^T x^*$.



Optimality conditions of LP

$$\min_x z = c^T x \text{ subject to } Ax = b,\ x \geq 0$$

- The Lagrangian function

$$\mathcal{L}(x,\lambda,s) = c^T x - \lambda^T(Ax - b) - s^T x$$

- The optimality conditions (KKT, see chap 12)

$$\begin{aligned} A^T\lambda + s &= c \\ Ax &= b \\ x, s &\geq 0 \\ x_i s_i &= 0 \end{aligned}$$

  - The last one is the *complementarity* condition

Primal-dual method

- The optimality conditions can be expressed as

$$F(x,\lambda,s) = \begin{pmatrix} A^T\lambda + s - c \\ Ax - b \\ XSe \end{pmatrix} = 0, \text{ for } x, s \geq 0$$

  - X=diag(x), S=diag(s) and e=(1,1,…,1)^T.
- F is a *nonlinear* equation: $\mathbb{R}^{2n+m} \to \mathbb{R}^{2n+m}$.
  - Solved by the Newton's method: (J the Jacobian of F.)

(x, λ, s)+=$\alpha(\Delta x, \Delta\lambda, \Delta s)$, $J(x,\lambda,s)\begin{pmatrix} \Delta x \\ \Delta\lambda \\ \Delta s \end{pmatrix} = -F(x,\lambda,s)$

Jacobian of *F*

- Let $F(x,\lambda,s) = \begin{pmatrix} A^T\lambda + s - c \\ Ax - b \\ XSe \end{pmatrix} = \begin{pmatrix} r_c \\ r_b \\ r_{XS} \end{pmatrix}$
- The Jacobian $J(x, \lambda, s)$ of $F(x, \lambda, s)$ is

$$J = \begin{pmatrix} \nabla_x r_c & \nabla_\lambda r_c & \nabla_s r_c \\ \nabla_x r_b & \nabla_\lambda r_b & \nabla_s r_b \\ \nabla_x r_{XS} & \nabla_\lambda r_{XS} & \nabla_s r_{XS} \end{pmatrix} = \begin{pmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{pmatrix}$$

Solving the linear systems

- The Newton's direction is obtained by solving

$$\begin{pmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta \lambda \\ \Delta s \end{pmatrix} = \begin{pmatrix} -r_c \\ -r_b \\ -r_{XS} \end{pmatrix}$$

- S is an artificial variable. Let $D = S^{-1/2} X^{1/2}$

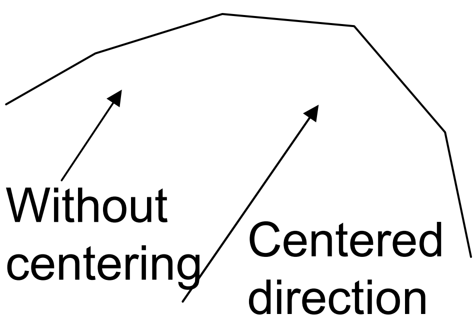
$$\begin{pmatrix} -D^{-2} & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta \lambda \end{pmatrix} = \begin{pmatrix} -r_c + X^{-1} r_{XS} \\ -r_b \end{pmatrix}$$

$$\Delta s = -X^{-1} r_{XS} - X^{-1} S \Delta x$$

  - The left hand side is symmetric, but indefinite.

Centering

- In practice, $r_{XS}=XSe-\sigma\mu e$ rather than $r_{XS}=XSe$.
- Duality measure: $$\mu = \frac{1}{n}\sum_{i=1}^{n} x_i s_i = \frac{x^T s}{n}$$
- Centering parameter $\sigma \in [0,1]$:
  - Decreasing as approaching solutions
  - This is related to the barrier method (chap 15)



The path following algorithm

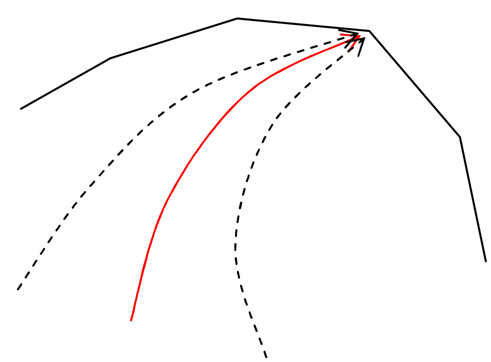
1. Given (x_0, λ_0, s_0) with $(x_0, s_0) > 0$
2. For $k = 0, 1, 2 \ldots$ until $(x_k)^T s_k < \varepsilon$
   a) Choose $\sigma_k \in [0,1]$ and let $\mu_k = (x_k)^T s_k / n$.
   b) Solve the Newton's direction $(\Delta x_k, \Delta\lambda_k, \Delta s_k)$
   c) Set $(x_{k+1}, \lambda_{k+1}, s_{k+1}) = (x_k, \lambda_k, s_k) + \alpha_k(\Delta x_k, \Delta\lambda_k, \Delta s_k)$, where α_k is chosen to make $(x_{k+1}, s_{k+1}) > 0$

Central path and neighborhood

- Feasible set $\mathbf{F}^0=\{(x,\lambda,s) \mid Ax=b,\ A^T\lambda+s=c,\ (x,s)>0\}$
- Point $(x_\tau,\lambda_\tau,s_\tau)\in\mathbf{F}^0$ is on the *central path C* if $x_\tau(i)s_\tau(i)=\tau$ for all $i=1,2,\ldots,n$.
- An example of the neighborhood of the central path

$$\mathbf{N}_{-\infty}(\circ) = \{(x,\lambda,s)\in\mathbf{F}^0 | x_i s_i \geq \circ\,\mu, i=1,2,\ldots,n\}$$

  - μ is the duality measure (two slides before)
  - Typically $\gamma=10^{-3}$.



Long-step path-following

1. Given $(x_0,\lambda_0,s_0)\in\mathbf{N}_{-\infty}$ and γ, σ_{min}, σ_{max}.
2. For $k = 0, 1, 2$ …until $(x_k)^T s_k < \varepsilon$
   a) Choose $\sigma_k\in[\sigma_{min},\sigma_{max}]$ and let $\mu_k=(x_k)^T s_k/n$.
   b) Solve the Newton's direction $(\Delta x_k, \Delta\lambda_k, \Delta s_k)$
   c) Set $(x_{k+1}, \lambda_{k+1}, s_{k+1})=(x_k, \lambda_k, s_k) + \alpha_k(\Delta x_k, \Delta\lambda_k, \Delta s_k)$, where α_k is chosen to make $(x_{k+1}, \lambda_{k+1}, s_{k+1})\in\mathbf{N}_{-\infty}$.

Complexity

- The long step following method can converge in $O(n\log(1/\varepsilon))$ iterations (Theorem 14.4)
- The most effective interior point method for LP is the predictor-corrector algorithm (Mehrotra 1992)