# CS5321 Numerical Optimization

13 Linear Programming: The Simplex Method

# The standard form

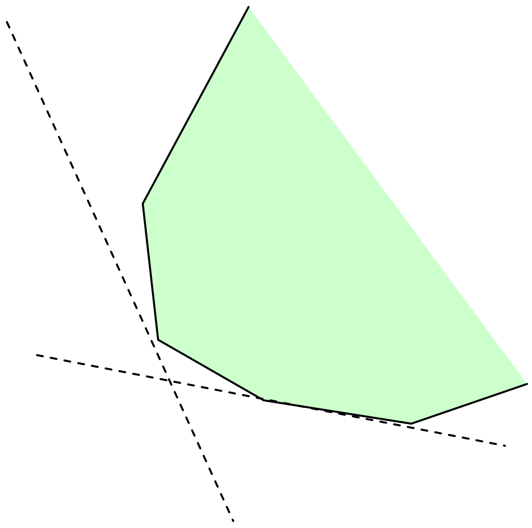
- The standard form of linear programming is

$$\text{Min}_x z = c^T x \text{ subject to } Ax = b,\ x \geq 0$$

  - Matrix $A$ is an $m \times n$ matrix, where $m$ is the number of constraints and $n$ is the number of variables.
  - We assume $A$ has full row rank and $m \leq n$.
  - For $Ax \geq b$, add *slack variables*. $Ax + z = b$, $z \geq 0$.
  - For $Ax \leq b$, subtract slack variables $Ax - z = b$, $z \geq 0$.

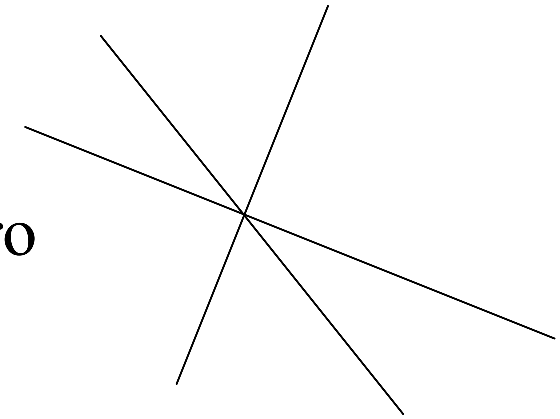
# Geometry of LP

- Feasible region $\Omega$: the set of all feasible points
  - If $\Omega$ is empty, LP has no solution. (infeasible)
  - If $\Omega$ is nonempty, it is convex.
- If the object function is unbounded on $\Omega$, LP has no solution.
- If LP is bounded and feasible, it can have either one or infinity many solutions.



# Basic feasible point

- A point x is a basic feasible point (or a vertex of feasible polytope) if it is feasible and is not a linear combination of any other feasible points.
- If LP has solutions, at least one solution is a basic feasible solution. (Theorem 13.2)
- At a basic feasible point, at least $n-m$ variables are zero.
  - The case it has more than $n-m$ zero variables is called *degenerate*.



# Basic variable and basis matrix

- At a basic feasible point, variables that can be uniquely determined is called *basic* variables.
  - The other are called *nonbasic* variables. (set to zero.)
  - Let $\mathcal{B}$, $\mathcal{N}$ be the index sets for basic/nonbasic variables.
- Variable $x$, $c$, and $A$ can be rearranged according to basic/nonbasic variables.

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}, c = \begin{pmatrix} c_B \\ c_N \end{pmatrix}, A = \begin{pmatrix} B & N \end{pmatrix}$$

  - $B$, an $n \times n$ nonsingular matrix, is called the *basis matrix*

# Simplex multiplier

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}, c = \begin{pmatrix} c_B \\ c_N \end{pmatrix}, A = \begin{pmatrix} B & N \end{pmatrix}$$

- Since $x_N$=0 at a basic feasible point
- Object function: $z = c^T x = c_B^T x_B + c_N^T x_N = c_B^T x_B$
- Constrains: $Ax = Bx_B + Nx_N = Bx_B = b$
- The basic variables are $x_B = B^{-1}b$ and therefore the object function $z = c_B^T x_B = c_B^T B^{-1} b$
- The simplex multiplier is $\lambda = (c_B^T B^{-1})^T = B^{-T} c_B$

# Pricing

- Object function $z = c^T x = c_B^T x_B + c_N^T x_N$ could be decreased by changing nonbasic variables, $x_N$.

  $Ax = Bx_B + Nx_N = b$, $\quad x_B = B^{-1}b - B^{-1}Nx_N$

  $z = c_B^T x_B + c_N^T x_N = c_B^T B^{-1} b + (c_N^T - c_B^T B^{-1} N) x_N$
- Plug in $\lambda = (c_B^T B^{-1})^T$, $z = c_B^T B^{-1} b + (c_N^T - \lambda^T N) x_N$
- The vector $s_N = c_N - N\lambda$ is called pricing.
  - If $s_N(i)<0$, $z$ can be decrease by increasing $x_N(i)$.
  - If all $s_N(i)\geq 0$, the optimal solution is founded.

# The ratio test

- Select $q$ s.t. $s_N(q)<0$ is the smallest element in $s_N$ and increase $x_N(q)$ until one element in $x_B$, say $x_B(p)$, becomes zero. (How to find $p$?)

$$\begin{aligned} x_B &= B^{-1}b - B^{-1}Nx_N & \text{(previous slide.)} \\ &= B^{-1}b - B^{-1}N(:,q)x_N(q) & (x_N = 0 \text{ except } x_N(q).) \end{aligned}$$

  - Need $x_B(i) \geq 0$ for all $i$. If $B^{-1}N(:,q)(i)<0$, then $x_B(i)>0$
  - Only need to consider $i$ for $B^{-1}N(:,q)(i)\geq 0$
    - What if no such $i$? the unbounded case

$$p = \arg\min_{i=1..m} \left\{ \frac{(B^{-1}b)(i)}{(B^{-1}N(:,q))(i)} \middle| (B^{-1}N(:,q))(i) \geq 0 \right\}$$

# Pivoting

- Exchange $p$ and $q$ in $\mathcal{B}$,$\mathcal{N}$ and update $x_B$, $x_N$ and $B$.
- Let $d = (B^{-1}N(:,q)), \circ = (B^{-1}b)(p)/d(p)$
  - Update $x_B = x_B - \gamma d$ and $x_N(q) = \gamma$
- Update $B$: replace $B(p)$ with $N(q)$ (How about $B^{-1}$?)
  - It is a rank-1 update. Let $B^+$ be the updated one.

$$B^+ = B + (N(:,q) - Be_p)e_p^T$$

  - The Sherman-Morrison formula

$$(B^+)^{-1} = B^{-1} - \frac{(B^{-1}N(:,q) - e_p)e_p^T B^{-1}}{1 + e_p^T(B^{-1}N(:,q) - e_p)}$$

# The simplex method

While (true)

1. Given $\mathcal{B},\mathcal{N}$. $x_B = B^{-1}b$, $x_N = 0$ (Basic feasible point)
2. $\lambda = B^{-1}c_B$, $s_N = c_N - N^T\lambda$ (Simplex multiplier, pricing)
3. If $s_N \geq 0$, stop (Found an optimal solution)
4. Select $q$ s.t. $s_N(q) < 0$, and solve $Bd = N(:,q)$
5. If $d \leq 0$, stop (Unbounded case)
6. Compute $[\gamma, p] = \min_{d(i)>0} x_B(i)/d(i)$ (Ratio test)
7. Update $x_B = x_B - \gamma d$ and $x_N(p) = \gamma$ (Pivoting)
8. Exchange $p$ and $q$ in $\mathcal{B},\mathcal{N}$ and update matrix $B$.

# Remaining problems

- How to find the initial basic feasible point?
  - Two phase algorithm: add more slack variables to make the trivial point (0,…,0) feasible, and solve it until all additional slack variables become zero.
- How to resolve the degenerate case?
  - In degenerate case, the algorithm might pivot the same p and q repeatedly.
  - Perturb the constraints to avoid the degenerate case.

# Complexity

- In each iteration, the most time consuming task is pricing, ratio test and update $B$. O($mn$)
- The number of iterations is less than or equals to the number of basic feasible points, which is

$$\binom{n}{m} = \frac{n!}{(n-m)!m!}$$

  - The worst case time complexity is **exponential**.
    - Try $n$=2$m$. The number of iterations > $2^m$.
  - But practically, it terminates in $m$ to 3$m$ iterations.
    - Average case analysis and smoothed analysis.